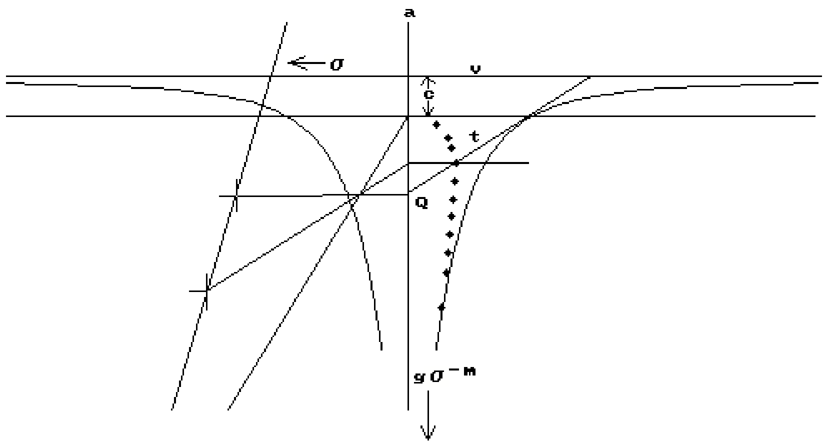
PIVOT TRANSFORM PROFILE



Consider a watery vortex with its invariant plane **v** a fractional distance c above the lower invariant plane of the transformation, generated by two ranges one on **v** with parameter σ and the other on the vertical axis **a** with parameter $g\sigma^{-m}$ where $-(\lambda_v+1) = m > 0$, m and g being constants. A tangent **t** meets **a** in Q, and we postulate a tangent plane to the vortex through **t** orthogonal to the plane (**v,a**).

A point of the range on **v** has coordinates $\{\sigma, c\}$, and its mate on **a** $\{0, c-g\sigma^{-m}\}$, so the tangent joining them is

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} \quad giving \quad \frac{x-\sigma}{-\sigma} = \frac{y-c}{\left(c-g\sigma^{-m}\right)-c}$$

i.e.

$$x = \frac{\sigma^{m+1}\left(y-c\right)}{g} + \sigma \tag{1}$$

Now from Lawrence Edwards the fractional height of the horizontal plane containing the pivot point is

$$h = \frac{k_2 h'}{k_2 h' + k_1\left(1-h'\right)} \tag{2}$$

where

$$h' = y = c - g\sigma^{-m}, \; k_1 = \sin^2\theta_2, \; k_2 = \sin^2\theta_1$$

noting for later

$$\sigma = \left[\frac{g}{c-h'}\right]^{1/m} \tag{3}$$

and

$$\tan\theta_1 = -\frac{\lambda+1}{2\,\epsilon\lambda} \; ; \; \tan\theta_2 = \frac{\lambda+1}{2\epsilon} \tag{4}$$

for the bud transform.

The coordinates of the pivot point for a profile, where we ignore the rotational parameter of the bud transform, are (x,h) where x is given by substituting for y in (1) for a given σ.

It should be noted that the rotation is through the fixed angle (θ_2-θ_1), and also implies a magnification of the above form by sec(θ_2-θ_1) as the pivot point lies on the line where the tangent plane intersects the profile plane. As (θ_2-θ_1) is constant we ignore it as we are interested in the shape. The above construction also implies that $\varepsilon > 0$ as $\varepsilon = 0$ reduces the radius to zero. This is because then all tangent planes are turning about the axis, so their pivot points all lie on the axis.

If we are fitting a pivot profile of known parameters to a curve, we may select h, find $y=h'$ from the inverse of (2), and then σ from (3), giving the corresponding x using (1). We may further find the rms error for the whole profile and optimise by iteration. The inverse of (2) is

$$h' = \frac{k_1 h}{k_1 h + k_2 (1-h)} \tag{5}$$

More difficult is to optimise the parameters λ, ε, *c, m* and *g* of the profile to fit a given curve as they are not 'orthogonal', and a linearised approach using matrices is impractical other than with known small integral values of *m*.

<u>Fitting a Curve</u>

A point on the curve is at the height h, so we substitute h for y in (1) to give the corresponding value of x:

$$x = \frac{\sigma^{m+1}(h-c)}{g} + \sigma = \left(\frac{g}{c-f}\right)^{m+1/m}\left(\frac{h-c}{g}\right) + \left(\frac{g}{c-f}\right)^{1/m} = \left(\frac{g}{c-f}\right)^{1/m}\left(\frac{h-f}{c-f}\right)$$

where $\quad f = f(h) = h'$, given by (5).

so that $\quad x^m (c-f)^{m+1} = g(h-f)^m \qquad (6)$

Thus $\quad \dfrac{c-f_1}{c-f_2} = \left(\dfrac{x_2(h_1-f_1)}{x_1(h_2-f_2)}\right)^{m/m+1}$ for two points (x_1,h_1) and (x_2,h_2).

so

$$c = \frac{f_1\{x_1(h_2-f_2)\}^{\mu} - f_2\{x_2(h_1-f_1)\}^{\mu}}{\{x_1(h_2-f_2)\}^{\mu} - \{x_2(h_1-f_1)\}^{\mu}} \tag{7}$$

where μ=m/(m+1).
Then given (x_3,h_3)

$$c = \frac{f_1\{x_1(h_2-f_2)\}^{\mu} - f_2\{x_2(h_1-f_1)\}^{\mu}}{\{x_1(h_2-f_2)\}^{\mu} - \{x_2(h_1-f_1)\}^{\mu}} = \frac{f_1\{x_1(h_3-f_3)\}^{\mu} - f_3\{x_3(h_1-f_1)\}^{\mu}}{\{x_1(h_3-f_3)\}^{\mu} - \{x_3(h_1-f_1)\}^{\mu}}$$

i.e.

$$\left[f_1\{x_1(h_2-f_2)\}^{\mu} - f_2\{x_2(h_1-f_1)\}^{\mu}\right]\left[\{x_1(h_3-f_3)\}^{\mu} - \{x_3(h_1-f_1)\}^{\mu}\right]$$

$$=\left[f_1\left\{x_1\left(h_3-f_3\right)\right\}^{\mu}-f_3\left\{x_3\left(h_1-f_1\right)\right\}^{\mu}\right]\left[\left\{x_1\left(h_2-f_2\right)\right\}^{\mu}-\left\{x_2\left(h_1-f_1\right)\right\}^{\mu}\right]$$

& simplifying

$$\left(f_3-f_1\right)\left\{x_1x_3\left(h_1-f_1\right)\left(h_2-f_2\right)\right\}^{\mu}+\left(f_1-f_2\right)\left\{x_1x_2\left(h_1-f_1\right)\left(h_3-f_3\right)\right\}^{\mu}+\left(f_2-f_3\right)\left\{x_2x_3\left(h_1-f_1\right)^2\right\}^{\mu}=0 \quad (8)$$

which is an equation in $k=k_2/k_1$ (c.f. f(h)) if we assume a value for m, or *vice versa*. Given m we may solve for k using bisection; if m is also unknown we may use double-bisection.
The danger of division by zero during the bisection requires (8) to be expressed as it is without division. Then we calculate c from (7) and substitute for c and k in (6) for (x_1, h_1) to give g, assuming we have converted the x_j and h_j to fractional values. How do we do the latter?

Other Relationships

We find the maximum value of x by differentiating (6) wrt $k=k_2/k_1$ which gives

$$\frac{d\left(x^m\right)}{dk}=\frac{g\,m(c-f)(h-f)^{m-1}\dfrac{d(-f)}{dk}-g(m+1)(h-f)^m\dfrac{d(-f)}{dk}}{(c-f)^{m+2}} \qquad (9)$$

df/dk=0 or h=f for h=0 or 1, which do not apply, so for a maximum

$$m(c-f)-(m+1)(h-f)=0$$

or

$$\mu(c-f)=(h-f) \qquad (10)$$

When h'=-∞, by (1) and inspection of the diagram σ=0 as q is at infinity downwards.
Then by (5) $k_1h+k_2(1-h)=0$ i.e.

$$h=\frac{k}{k-1} \qquad (11)$$

For x=0, by (6) either h=f, which by (5) implies h=0 or 1, h=0 giving the top of the pivot transform on the bottom invariant plane of the bud, and h=1 giving the top invariant plane of the bud. Or else by (1) σ=0 as for (11). By (1) the gradient dx/dy is

$$\frac{dx}{dy}=\frac{\sigma^{m+1}}{g} \qquad (12)$$

This is zero as σ=0, giving a tapered profile. If the profile is not tapered when x=0 then the pivot transform lies above the bottom invariant plane, crossing the axis at h=1 on the top invariant plane. Then h=1 so h'= 1 also, and by (3)

$$\sigma=\left[\frac{g}{c-1}\right]^{1/m}$$

which is a constant, so substituting in (12)

$$\frac{dx}{dy} = \left[\frac{g}{(c-1)^{m+1}} \right]^{\frac{1}{m}} \tag{13}$$

i.e. only one value is possible for a given set of parameters.
Now f=∞ when h is given by (11) , and from (6)

$$x^m \rightarrow \frac{g}{-f}$$

that is x=0 for this value of h, which provides another possible vertical axis crossing. An example is k=0.5 and h = -1, which is useful for fitting profiles as the scaling factor for converting y to h is not known. But taking h=-1 for the crossing point requires k=0.5, and then we find μ using bisection with (8) and then c from (7) and g from (6). The measured heights y are converted to their corresponding h values by dividing by s=y at the crossing point.

Tapered Profile

If x_3=0 then by (11) we have for the fully tapered case

$$h_3 = \frac{k}{k-1} \tag{T1}$$

which is the fractional height corresponding to y_3.

For the maximum value of x we substitute (11) in (6), and noting that g is a scaling factor which we now set to 1, we obtain

$$h - f = \frac{\mu^{(m+1)}}{x^m} = M$$

From (5)

$$h - f = h\left[\frac{(k-1)(1-h)}{h+k(1-h)} \right]$$

and from (T1)

$$k = \frac{h_3}{h_3 - 1} \tag{T2}$$

giving

$$h - f = h\left[\frac{\dfrac{(1-h)}{(h_3-1)}}{\left\{ h + \dfrac{h_3(1-h)}{(h_3-1)} \right\}} \right] = h\left[\frac{(1-h)}{h_3 - h} \right] = M$$

Now for some scaling factor s to convert actual to fractional heights, $h_3=y_3/s$ and h=y/s, giving

$$\frac{y}{s}\left[\frac{s-y}{y_3-y}\right]=M$$

so

$$s=\frac{y^2}{y+M\left(y-y_3\right)} \tag{T3}$$

which we recall is in terms of (x,y) for the maximum radius x of the profile.

Substituting y_3/s for h_3 in (T2) gives k.

c is then found from (6)

$$c=\left(\frac{h-f}{x}\right)^{\mu}+f=\left(\frac{M}{x}\right)^{\mu}+\frac{y}{y+k(s-y)}$$

which solves the problem, recalling that we assume we know m = $-(\lambda_v+1)$ and that it is valid to set g=1.

<u>Summary</u>

For a fully tapered form with a well-defined maximum we set g=1, μ=m/(m+1) for known m = $-(\lambda_v+1)$, and take y_3 to be the actual height at which the radius x_3 becomes zero. Then we solve

$$s=\frac{y^2}{y+M\left(y-y_3\right)}$$

for the scaling factor s which converts actual to fractional heights, where (x,y) is the point on the profile for which x is a maximum and $M=\frac{\mu^{(m+1)}}{x^m}$.

Then

$$k=\frac{k_2}{k_1}=\frac{y_3}{y_3-s}$$

$$f=\frac{y}{y+k(s-y)}$$

and

$$c=\left(\frac{M}{x}\right)^{\mu}+f$$

The theoretical profile is then plotted for a series of values of y using

$$x=\frac{\left(\frac{y}{s}-f\right)}{(c-f)^{\frac{1}{\mu}}}$$