PIVOT TRANSFORM PLOTTING

N.C. Thomas

Introduction

This is for simple programs to plot pivot transform profiles. The assumption is a "warm" context in which a range of projective geometry functions are available such as finding various incidence conditions. In particular the use of plane coordinates for finding where planes intercept each other or where they intercept given lines. In addition it is assumed the use of a semi-imaginary invariant tetrahedron is provided for (e.g. program PROJGEOM).

General Path Curves

First we will look at the general case of transforming a path curve. The general transformation matrix is

$$\begin{bmatrix} \varphi e^{i\theta} & & & \\ & \varphi e^{-i\theta} & & \\ & & \Lambda_2 & \\ & & & \Lambda_3 \end{bmatrix} \tag{1}$$

where the characteristic roots are Λ_0 .. Λ_3, but for the semi-imaginary case Λ_0 and Λ_1 are as shown with ϕ=the modulus and θ = the angle turned through. A general point z_i in space is such that z_0 is the horizontal homogeneous x-coordinate, z_1 the horizontal coordinate orthogonal to z_0, z_2 the vertical coordinate which is zero in the lower invariant plane, and z_3 the coordinate which is zero in the upper invariant plane. We use the symbol z to indicate that real points have complex coordinates in this reference system. If z_i lies in the lower invariant plane then z_2=0. The radial distance r of the point from the axis is calculated using the moduli $|z_i|$ of the coordinates i.e.

$$r=\sqrt{\left(\frac{|z_0|}{|z_3|}\right)^2+\left(\frac{|z_1|}{|z_3|}\right)^2}=\frac{\sqrt{|z_0|^2+|z_1|^2}}{|z_3|}$$

Transforming z_i gives $\{z_0e^{i\theta}, z_1e^{-i\theta}, 0, \Lambda_3 z_3\}$, and the radial distance is now

$$\frac{\sqrt{|\phi z_0 e^{i\theta}|^2+|\phi z_1 e^{-i\theta}|^2}}{\Lambda_3|z_3|}=\frac{\phi\sqrt{|z_0|^2+|z_1|^2}}{\Lambda_3|z_3|}=\frac{\phi}{\Lambda_3}r$$

so that the transform causes a radial expansion ϕ/Λ_3 for each angle θ turned through. A similar argument applies to the upper invariant plane giving

$$\begin{aligned}&\text{radial expansion in lower plane} = \phi/\Lambda_3\\ &\text{radial expansion in upper plane} = \phi/\Lambda_2\end{aligned} \tag{2}$$

This of course gives a logarithmic spiral in each invariant plane (as we have a constant ratio between the radial expansion and the angle turned through), which has the general equation

$$r = ae^{\theta \cot \alpha}$$

so going from r_1 to r_2 and using (2)

$$\frac{r_2}{r_1} = e^{(\theta_2 - \theta_1)\cot\alpha} = \frac{\phi}{\Lambda}$$

where $(\theta_2 - \theta_1)$ is our stepping angle θ. Hence

$$\tan\alpha = \frac{\theta}{\log\left(\frac{\phi}{\Lambda}\right)} \qquad (3)$$

giving the characteristic angle α for all path curve spirals in an invariant plane.

λ is defined as the negative ratio of the logs of (2) i.e.

$$\lambda = -\frac{\log\phi - \log\Lambda_2}{\log\phi - \log\Lambda_3} \qquad (4)$$

Thus

$$\log\phi = \frac{\lambda\log\Lambda_3 + \log\Lambda_2}{1+\lambda} \qquad (4a)$$

Normalising by dividing (1) by ϕ i.e. setting $\phi=1$ then gives

$$\lambda = -\frac{\log\Lambda_2}{\log\Lambda_3} \qquad (5)$$

ε is defined as ½ [$\log(m_1)+\log(m_2)$] where m_1, m_2 are the multipliers in the invariant planes given by (2), and recalling that for a bud one of m_1 m_2 is less than 1 so that one of the logs is negative, we have

$$\varepsilon = \frac{\log\Lambda_2 - \log\Lambda_3}{2\theta} \qquad (6)$$

defining ε per unit radian turned.

Solving (4) and (5) for $\log(\Lambda_2)$ and $\log(\Lambda_3)$ gives

$$\log\Lambda_2 = \frac{2\,\varepsilon\theta\lambda}{1+\lambda} \qquad (7)$$

and

$$\log \Lambda_3 = -\frac{2\varepsilon\theta}{1+\lambda} \tag{8}$$

Thus given λ, ε and θ we may find Λ_2 and Λ_3 and then ϕ from (4a) and hence set up the transformation (1).

We set up the basic transform upon which the pivot transform itself is based using the appropriate λ, ε and θ.

To transform a general path curve we set up another transform **T** in the same way using λ_1, ε_1 and θ_1. Given any starting plane $\mathbf{u_0}$, we transform it using $\mathbf{T^{-1}}$ i.e. $\mathbf{u_1=T^{-1}u_0}$ and then find the pivot point as follows:

1. Find the point Q in which $\mathbf{u_1}$ meets the vertical axis {0,0,0,0,1,0}

2. Calculate the fractional height h of the horizontal plane in which the pivot point lies from the fractional height h' of Q using

$$h = \frac{k_2 h'}{k_2 h' + k_1 (1-h')} \tag{9}$$

where $k_1 = \sin^2\alpha_2$ and $k_2 = \sin^2\alpha_3$, α_2 being the angle given by (3) for the upper invariant plane, and α_3 for the lower one.

3. Find the line **l** in which $\mathbf{u_1}$ intersects the upper invariant plane {0,0,0,1} and hence the perpendicular distance d of the upper invariant point {0,0,1,0} from **l**. Then the radial distance r of the pivot point from the axis is

$$r = \frac{h.d.\sin(\alpha_2+\alpha_3)}{\sin^2\alpha_2} = h.d.k \tag{10}$$

defining k accordingly (see Reference 1).

These operations are not expanded further as the program provides the required functions.

For plotting a profile we use r and h, ignoring θ_1. Otherwise plot {r.cos(θ_1), r.sin(θ_1), h} for the twisted curve. Repeat this for $\mathbf{u_2=T^{-1}u_1}$, $\mathbf{u_3=T^{-1}u_2}$ etc. to obtain the points on the curve.

It is interesting that profiles depend only upon λ_1 and the product $\varepsilon_1\theta_1$ c.f. (1) (7) and (8), because we ignore the rotation produced by θ_1. For a surface of circular cross-section produced by rotating a twisted pivot curve through 360° the vertical profile will be what we have referred to as the profile. Hence we use the profile when working with surfaces.

This handles general path curves where both the invariant planes of the basic transform and those of the path curve being transformed are accessible. For invariant planes at infinity we must proceed otherwise.

Cosmic Vortex

For a cosmic vortex we have the height H=∞ and the equation

$$h = \left(\frac{h_0}{r_0^{1+\lambda}}\right) r^{1+\lambda} = kr^{1+\lambda} \tag{11}$$

c.f. equation (15) in Reference 2. Then

$$\frac{dh}{dr} = (1+\lambda) kr^{\lambda} = m \tag{12}$$

gives the gradient of the tangent at a point (r,h). The tangent line is thus

$$y - \bar{y} = m(x - \bar{x}) + c \tag{13}$$

where $(\bar{x}, \bar{y})$ are the coordinates of the vortex origin wrt the invariant tetrahedron. At a point (r_1,h_1) on the curve we have

$h_1 = mr_1 + c = \{(1+\lambda)kr_1^{\lambda}\}r_1 + c = (1+\lambda)h_1 + c$, so $c = -\lambda h_1$. Thus the tangent line is

$$m(x - \bar{x}) - y + \bar{y} - \lambda(y_1 - \bar{y}) = 0$$

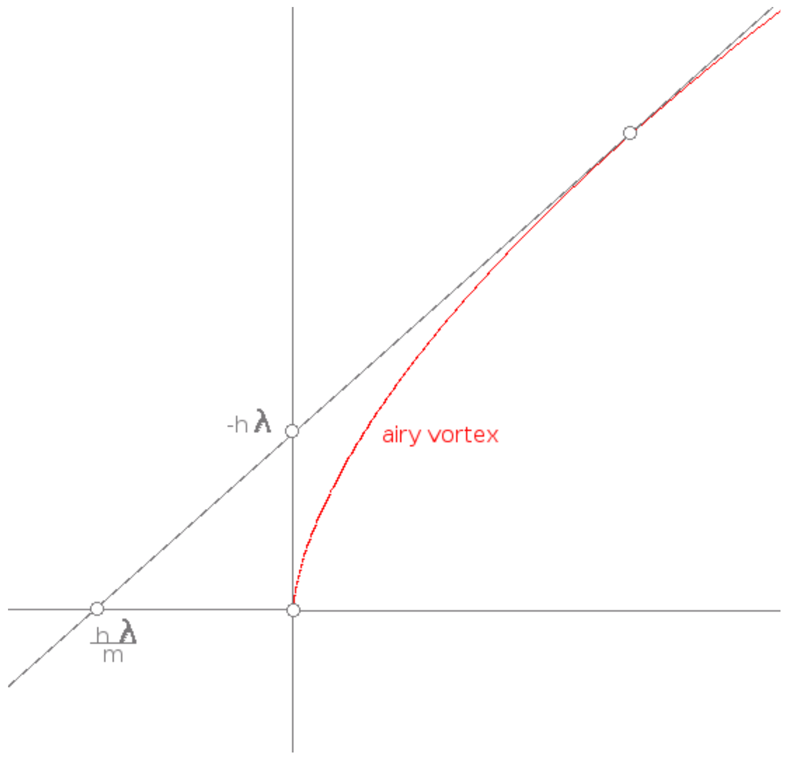
i.e. (14)

$$mx - y + (1+\lambda)\bar{y} - \lambda y_1 - m\bar{x} = 0$$

or

$$u_0 x + u_1 y + u_3 = 0 \tag{15}$$

where $u_0 = m$, $u_1 = -1$ and $u_3 = (1+\lambda)\bar{y} - \lambda y_1 - m\bar{x}$. These are the coordinates of the tangent plane to the profile if we set $u_2 = 0$ so that the third coordinate is free to be any value, which extends the tangent line to a plane orthogonal to the diagram (e.g. for an air vortex):



If $-1 < \lambda < 0$ then $h = 0 \leftrightarrow r = 0$ in (11) giving an air vortex, while if $\lambda < -1$ we obtain a water vortex. The intercepts are given by setting first $r = x - \bar{x} = 0$ and then $h = y - \bar{y} = 0$ in (13) with $c = -\lambda h_1$.

Thus given h_1 we have

from (11) $r_1 = \left(\frac{h_1}{k}\right)^{\frac{1}{1+\lambda}}$ which substituted in (12) gives

$$m = (1+\lambda) k^{\frac{1}{1+\lambda}} h_1^{\frac{\lambda}{1+\lambda}} \tag{16}$$

and then $u_0 = m$, $u_1 = -1$, $u_2 = 0$, $u_3 = (1+\lambda)\bar{y} - \lambda y_1$ (17)

noting that we are using homogeneous coordinates. Apply steps 1 to 3 of the previous section to find the pivot point in **u**.

Repeat this for various values of h to obtain the profile.

General Forms

The above solutions show how we proceed for any surface: find the equation $ax+by+c=0$ of the tangent in the profile plane; then $u_0 = a$, $u_1 = b$, $u_2 = 0$, $u_3 = c$. Plot the curve by varying that plane systematically. Thus for an ellipse we have

$$\frac{(x-\bar{x})^2}{a^2} + \frac{(y-\bar{y})^2}{b^2} = 1 \tag{18}$$

with a tangent at (x_1, y_1):

$$\frac{(x-\bar{x})(x_1-\bar{x})}{a^2}+\frac{(y-\bar{y})(y_1-\bar{y})}{b^2}=1 \tag{19}$$

For a point on the vertical axis of the conic $x=\bar{x}$ so $y_1-\bar{y}=\frac{b^2}{Y}$ if (X,Y) is a point on the conic referred to coordinates with its centre at the origin. Also

$$x_1-\bar{x}=a\sqrt{\frac{(y_1-\bar{y})^2}{b^2}-1}=a\sqrt{\frac{b^2}{Y^2}-1} \tag{20}$$

Then rearranging (19) for the tangent line we have

$$\frac{(x_1-\bar{x})}{a^2}x+\frac{(y_1-\bar{y})}{b^2}y+\left\{-\frac{(x_1-\bar{x})}{a^2}\bar{x}-\frac{(y_1-\bar{y})}{b^2}\bar{y}-1\right\}=0$$

and substituting from (20) and using b^2/Y we get the plane coordinates

$$u_0=\frac{1}{a}\sqrt{\frac{b^2}{Y^2}-1};\;\; u_1=\frac{1}{Y};\;\; u_2=0;\;\; u_3=-(u_0\bar{x}+u_1\bar{y}+1)$$

Hence given a, b, $(\bar{x},\bar{y})$ we vary Y (the height of the tangent point wrt the semi-major axis) to obtain the pivot profile from the planes **u**.

Plotting Complete Curve

Especially in the case of the cosmic vortex it can be difficult to obtain the complete profile. If we select the fractional height h of the horizontal plane in which we want a pivot point then from (9)

$$h'=\frac{k_1 h}{k_1 h+k_2(1-h)} \tag{21}$$

gives us the fractional height of Q where the tangent plane intersects the axis, and thus for the surface being transformed the height of Q is $h'H-\bar{y}$ in its own coordinate system, where H is the distance between the invariant points on the axis. Thus $h'H-\bar{y}=-m\bar{x}+c$ for the tangent line to the profile as x = 0 for Q. e.g. for the cosmic vortex $c=-\lambda h_1$ from (14) where (r_1, h_1) is the point of tangency, and assuming $\bar{x}=0$ we have $h_1=-c/\lambda=(\bar{y}-h'H)/\lambda$ provided m is not infinite, and hence using (16) we have

$$m=(1+\lambda)k^{\frac{1}{1+\lambda}}\left(\frac{\bar{y}-h'H}{\lambda}\right)^{\frac{\lambda}{1+\lambda}} \tag{22}$$

giving $u_0 = m$, and

$$u_3=(1+\lambda)\bar{y}-\lambda y_1=\bar{y}-\lambda h_1=h'H \tag{23}$$

We now know **u** for the plane with its pivot point at fractional height h, and thus find r from (10) to give the profile point. By varying h from 0 to 1 we obtain points on the complete profile between the invariant planes.

Whether a complete profile is actually obtained depends upon the choice of $\bar{y}$, as a negative value of $\left(\frac{\bar{y}-h' H}{\lambda}\right)$ in (23) results in a complex value of m and hence no real tangent plane, so the pivot transform is then genuinely open at the bottom or top or both. If $\bar{y} < 0$ this is less likely as $\lambda < 0$, but on the other hand if $h' < 0$ complex values may still arise.

<u>References</u>

1. *Pivot Transformations*, N.C. Thomas.
2. *Practical Path Curve Calculations*, N.C.Thomas.