A New Approach to Pivot Transforms

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Lawrence Edwards first described pivot-transforms (Edwards 1993). Given a linear transformation **T** of 3-space, every plane is being moved by **T** and in each such plane there is in general one point P about which a given such plane is "pivotting" i.e. that point is momentarily motionless. Edwards called such points "pivot-points". This follows because every plane is the osculating plane of some path curve generated by **T**, and its point of contact is momentarily invariant. The invariant curves determined by **T** are called path-curves and form a complete covering of space, so every point of space contains an osculating plane of a path curve and conversely every plane of space is such an osculating plane (excluding the reference tetrahedron). This one-to-one relation established between the points and planes of space will be proved below. Edwards suggested that a surface can be transformed by taking all its tangent planes and finding their pivot points, yielding a generally distinct point-wise surface which is said to be the pivot-transform of the first.

He analysed the situation for the case when the invariant tetrahedron of **T** is semi-imaginary i.e. only two of its invariant planes, two of its invariant points and two of its invariant lines are real. This paper will derive a strikingly simple expression for a pivot transform in the general case i.e. valid for any linear transformation. As this is accomplished using parametric coordinates for path curves they will be described again for convenience. Familiarity with linear geometric transforms of 3-space and their path curves, as well as the use of homogeneous coordinates, will be assumed (Edwards 1993).

One advantage of the new approach is that inverse pivot-transforms can readily be found, as explained below.

Parametric Equations for Path Curves

The homogeneous parametric equations of a path curve with t as parameter are (Thomas 2001):

$$\{\ x = k_1 t^p :\ y = k_2 t^q :\ z = k_3 t^r :\ t\ \} \tag{1}$$

This is verified as follows. Let the point $P_1(t_1)$ be transformed into $P_2(t_2)$ by the following linear transformation:

$$\begin{bmatrix} (t_2/t_1)^p & & & \\ & (t_2/t_1)^q & & \\ & & (t_2/t_1)^r & \\ & & & (t_2/t_1) \end{bmatrix} \begin{bmatrix} k_1 t_1^p \\ k_2 t_1^q \\ k_3 t_1^r \\ t_1 \end{bmatrix} = \begin{bmatrix} k_1 t_2^p \\ k_2 t_2^q \\ k_3 t_2^r \\ t_2 \end{bmatrix}$$

Setting $d=t_2/t_1$, any point P(t) of the curve is transformed by this into another point of the curve:

$$\begin{bmatrix} d^p & & & \\ & d^q & & \\ & & d^r & \\ & & & d \end{bmatrix} \begin{bmatrix} k_1 t^p \\ k_2 t^q \\ k_3 t^r \\ t \end{bmatrix} = \begin{bmatrix} k_1 (d\,t)^p \\ k_2 (d\,t)^q \\ k_3 (d\,t)^r \\ d\,t \end{bmatrix} \tag{2}$$

Thus (1) is a path curve of the linear transformation in (2), and clearly any value for d other than 0 may be chosen, all relating to the same system of path curves. Distinct curves have distinct values of the k_j.

For a general leading diagonal [\a b c d\] for the transform, p=log(a)/log(d), q=log(b)/log(d) and r=log(c)/log(d). If a b c and d are not all of the same sign then this fails for real coordinates referred to a real invariant tetrahedron. However that is a peculiar case involving a path curve with two branches, which can be handled by using the transformation [\$a^2$ b^2 c^2 d^2\] for one branch, and [\a b c d\] once and then [\$a^2$ b^2 c^2 d^2\] subsequently for the other. If the invariant tetrahedron is not fully real then the coordinates in (2) are not real, so with complex values for a b c and d no problem arises with the logs. In that case t must be complex and most naturally may be taken as $t=e^{-i\theta}$ where θ is the parameter.

The Cartesian coordinates for (1) are $x = k_1t^{p-1}$, $y = k_2t^{q-1}$, $z = k_3t^{r-1}$

or defining α=p-1 etc.: $x = k_1t^{\alpha}$, $y = k_2t^{\beta}$, $z = k_3t^{\gamma}$ (3)

Pivot Transform

Differentiating (3) with respect to t gives the direction of the tangent to the path curve as the vector

$$\{\alpha x/t, \beta y/t, \gamma z/t\} \tag{4}$$

at the point given by t.

Differentiating (3) twice wrt t yields

$$\frac{d^2x}{dt^2}=\frac{\alpha(\alpha-1)x}{t^2},\quad \frac{d^2y}{dt^2}=\frac{\beta(\beta-1)y}{t^2},\quad \frac{d^2z}{dt^2}=\frac{\gamma(\gamma-1)z}{t^2}$$

so that the direction of the binormal at t is the outer-product of this and the first derivative (Rutherford):

$$\left\{\frac{\beta\gamma(\gamma-\beta)}{t^3}yz,\ \ \frac{\gamma\alpha(\alpha-\gamma)}{t^3}zx,\ \ \frac{\alpha\beta(\beta-\alpha)}{t^3}xy\right\} \tag{5}$$

The binormal is orthogonal to the osculating plane, so its components in (5) are proportional to the homogeneous coordinates of the osculating plane itself (c.f. Appendix A), which is thus

$$\left\{\frac{\beta\gamma(\gamma-\beta)}{t^3}yz,\ \ \frac{\gamma\alpha(\alpha-\gamma)}{t^3}zx,\ \ \frac{\alpha\beta(\beta-\alpha)}{t^3}xy,\ \ w\right\} \tag{6}$$

for some w. Since the point {x,y,z,1} lies in this plane, w is given by

$$w=-\frac{xyz}{t^3}\{\beta\gamma(\gamma-\beta)+\gamma\alpha(\alpha-\gamma)+\alpha\beta(\beta-\alpha)\}$$

so (6) becomes

$$\left\{\frac{\gamma-\beta}{\alpha x},\ \frac{\alpha-\gamma}{\beta y},\ \frac{\beta-\alpha}{\gamma z},\ -\frac{\gamma-\beta}{\alpha}-\frac{\alpha-\gamma}{\beta}-\frac{\beta-\alpha}{\gamma}\right\} \tag{7}$$

This plane, being an osculating plane, has triple contact with the curve i.e. P{x y z 1} is momentarily fixed as t is varied, and is the pivot point of the osculating plane. Now (3) is for one path curve of the family, and the k_j determine the individual curve of the family which is in turn determined by α, β and γ. The pivot point occurs where a path curve of the family has the plane as its osculating plane. Thus given a tangent plane u_j to some surface, its pivot point is found by equating its components to (7) and solving for x, y and z:

$$u_0=\frac{\gamma-\beta}{\alpha x},\quad u_1=\frac{\alpha-\gamma}{\beta y},\quad u_2=\frac{\beta-\alpha}{\gamma z},\quad u_3=\frac{-\gamma-\beta}{\alpha}-\frac{\alpha-\gamma}{\beta}-\frac{\beta-\alpha}{\gamma} \tag{8}$$

so that

$$x=\frac{\gamma-\beta}{u_0\alpha},\quad y=\frac{\alpha-\gamma}{u_1\beta},\quad z=\frac{\beta-\alpha}{u_2\gamma} \tag{9}$$

For a discrete transformation such as (2) with leading diagonal [\a b c d\] it was shown that p=log(a)/log(d), q=log(b)/log(d) and r=log(c)/log(d), having defined α=p-1, β=q-1, γ=r-1, c.f. (3), so from (9)

$$x=\frac{\log(c/b)}{u_0\log(a/d)},\quad y=\frac{\log(a/c)}{u_1\log(b/d)},\quad z=\frac{\log(b/a)}{u_2\log(c/d)} \tag{10}$$

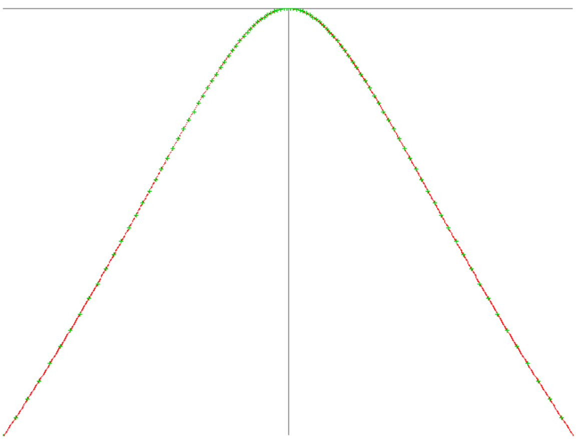
in Cartesian coordinates, and the pivot point in homogeneous coordinates is thus

$$x_j=\{X/u_0\ ,\ Y/u_1\ ,\ Z/u_2\ ,\ W/u_3\} \tag{11}$$

for the constants X Y Z defined according to (10), with W=-X-Y-Z from (8). Then $\Sigma_j x_j u_j=0$ for x_j to lie on u_j.

The pivot point is clearly uniquely determined for a given plane and conversely given any point one and only one plane has that point as its pivot point because the u_j are uniquely determined by (11), so a one-to-one correspondence between the planes and points of space is established by a pivot-transform (excluding the invariant tetrahedron).

The above derivation of (11) is valid for any tetrahedron of reference, so it is also valid for complex coordinates when the tetrahedron is not fully real. It is then convenient to calculate the pivot-transform in those coordinates and transform the result back to real coordinates to display the result (see Appendix B). The special case for a "semi-imaginary" invariant tetrahedron was treated by Lawrence Edwards (Edwards *op cit*), but (11) enables all cases to be handled. The following diagram shows a vertical cross-section of the pivot transform of a sphere determined using Edwards' method (in red) with that using (11) as green crosses:



Edwards found that for the semi-imaginary tetrahedron all planes intersecting the vertical axis in a fixed point Q{0 q 0 1} have their pivot-points in a fixed horizontal plane. Planes u_j intersecting the y-axis in a fixed point are such that the point {0 q 0 1} lies in them i.e. u_1=-u_3/q. Referring to (11), they have pivot points P{X/u_0, -Yq/u_3, Z/u_2, W/u_3} so that the Cartesian y-coordinate is -Yq/W i.e. it is constant. For a semi- or fully-imaginary tetrahedron Q has complex coordinates, and when referred back to real Cartesian coordinates the relationship between the y-coordinates of Q and the pivot-points P is not a simple proportionality. It is shown in Appendix C that for a semi-imaginary tetrahedron the pivot-points all have the y-coordinate

$$\frac{qhY}{q(Y+W)-Wh} \tag{12}$$

{0 h 0 1} being the point where the y-axis intersects the top invariant plane (c.f. (C3)). For a fully-imaginary tetrahedron (c.f. (C5)) the pivot-points have a y-coordinate

$$\frac{sq-r}{rq+s} \tag{13}$$

where Y=r+i.s and W=r-i.s .

Similar considerations apply, of course, to planes intersecting any other axis in a fixed point.

Given a general axial-pencil about the line $u_j+\lambda v_j$ for fixed planes u_j v_j, it follows from (11) that the coordinates of the pivot point are cubic expressions in λ and hence it traverses a twisted cubic, as shown by Edwards for a semi-imaginary tetrahedron.

Consider now the bundle of planes $u_j = \lambda s_j+\mu v_j+\nu w_j$ for fixed planes s_j v_j w_j. If λ is varied while holding μ and ν constant a twisted cubic is obtained as above. If then μ is varied while still holding ν fixed a set of twisted cubics is obtained which form a surface. This is because holding λ and ν fixed gives a twisted cubic intersecting all those arising when μ and ν were constant, and then also varying λ leads to the same result as initially. Thus the pivot-transform of a bundle is a cubic surface.

If the plane contains the origin {0 0 0 1}, then (11) is expressed as

$$\{Xu_3/u_0\ ,\ Yu_3/u_1\ ,\ Zu_3/u_2\ ,\ W\}$$

and now u_3=0. The pivot-point is then {0 0 0 W} i.e. the origin. This implies that no plane through the origin is the proper osculating plane of a path curve. Similarly for the other vertices of the invariant tetrahedron, and also all points on its edges (see also Appendix C for pivot points on the y-axis). Thus the pivot-transforms of surfaces with tangent planes through the origin contain the origin. The cubic surface of the bundle in the origin has thus degenerated to a point. A surface with a tangent plane parallel to an axis has the corresponding u_j=0 so its pivot-transform contains at least one point at infinity.

Developables

A developable is the polar of a curve i.e. it is a single-parameter set of planes. It possesses a *cuspidal edge* which is in general a twisted curve, and its planes are the osculating planes of that curve. A path curve of **T** is the cuspidal edge of a developable, and the pivot transform of that developable is clearly the path curve itself.

Self-Corresponding Surfaces

The conditions for a surface to coincide with its pivot transform will now be investigated. The first solution comes from a consideration of asymptotic curves in a surface. Such curves have their osculating planes tangential to the surface (c.f. any text on vector methods or differential geometry, such as Rutherford 1962), so taking a vortex surface **V** which is an invariant surface of the underlying linear transform **T**, the asymptotic curves of **V** are path curves (Thomas 2009). The pivot points of the osculating planes of those curves are thus their points of tangency and so lie in **V**. Hence **V** is a self-corresponding surface. As a surface must have negative total curvature to contain asymptotic lines, bud surfaces cannot be self-corresponding in this way i.e. **T** must have vortical path curves.

The second conceivable case would be for a surface **S** to have every one of its tangent planes coinciding with an osculating plane of a path curve at its point of tangency. But that means every tangent plane is tangential to an invariant surface of **T**, so **S** would itself be such a surface i.e. no new possibilities arise.

For cases other than the above solution based on coincident tangent planes and their pivot-points, every tangent plane of a surface **S** must contain its pivot-point elsewhere on the surface than at its point of tangency. No solution has yet been found for this.

<u>Polar-Conjugate Pairs of Pivot Transforms and Inverse Transforms</u>

From (11) $x_0=X/u_0$, so $u_0=X/x_0$, and similarly for the other coordinates. Thus if the coordinates $\{x_0\ x_1\ x_2\ x_3\}$ of the pivot-point are re-interpreted as the coordinates of a plane then the coordinates $\{u_0\ u_1\ u_2\ u_3\}$ are the coordinates of its pivot-point. Now when a point $X\{x_0\ x_1\ x_2\ x_3\}$ is re-interpreted in this way, the plane $V\{x_0\ x_1\ x_2\ x_3\}$ is the polar plane of X with respect to the imaginary unit sphere. Hence if a surface **P** is the pivot-transform of a surface **S**, then reciprocally the surface **S'** polar to it with respect to the imaginary unit sphere is the pivot transform of the surface **P'** polar to **P**.

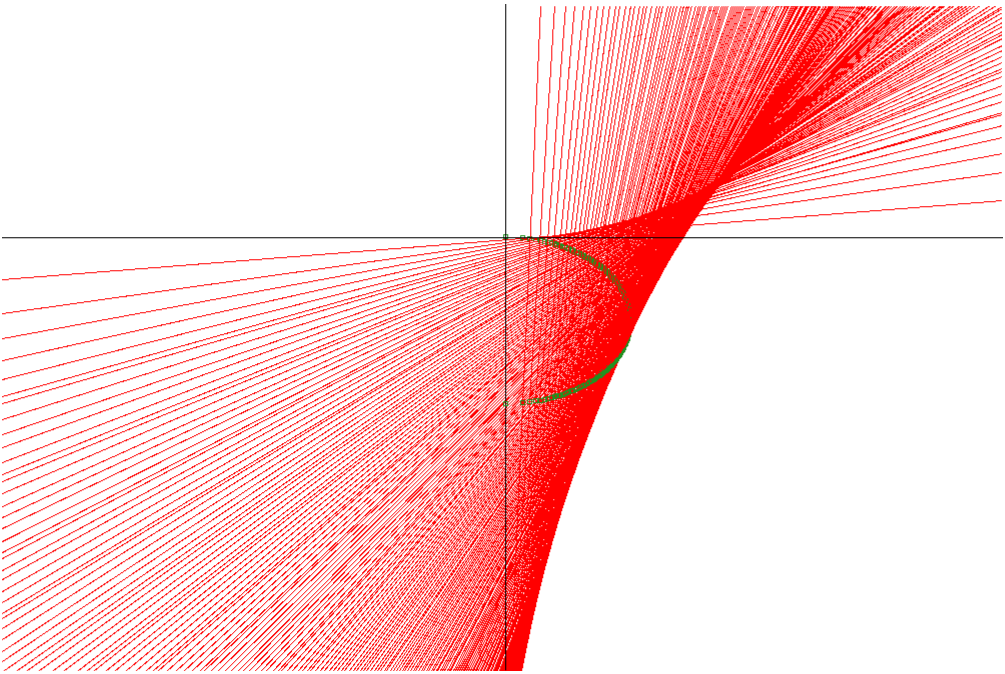
Thus to find the surface **S** which transforms into a given surface **P**, first construct the polar surface **P'** of **P** with respect to the imaginary unit sphere, find the pivot_transform **S'** of **P'** and finally find the polar **S** of **S'**. In practice this means:

1. Interpret the coordinates of the points of **P** as coordinates of planes, and find their transforms;

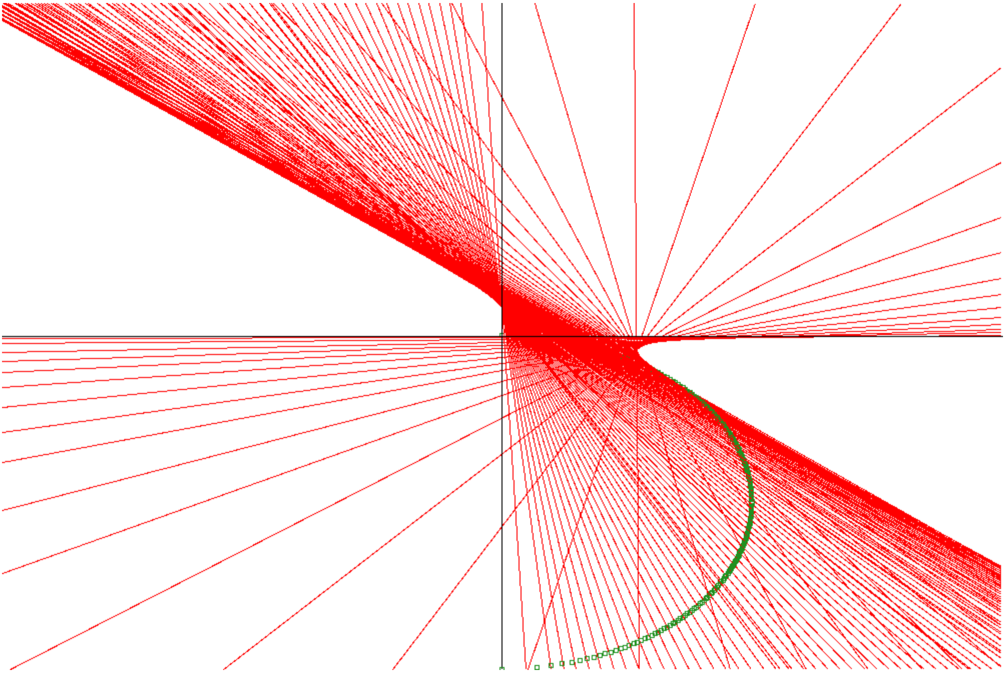
2. Interpret the coordinates of the resulting points of **S'** as coordinates of the tangent planes of **S**.

Notice the subtle difference from simply transforming the points of **P**, which lies in the fact that their coordinates must be transformed into the reference system like planes, not like points.

The following diagram shows, using this mehod, the inverse transform (red tangents) of a sphere for a semi-imaginary tetrahedron. For clarity only one semi-circle is shown, suppressing the rotation about the y-axis:



The following diagram shows the inverse transform of a sphere,with the same provisos as above, for a fully imaginary tetrahedron:



References

The Vortex of Life, Lawrence Edwards, Floris Press, Edinburgh 1993.
Vector Methods, D.E. Rutherford, Oliver and Boyd, Edinburgh and London 1962.
Asymptotic Path Curves, N.C. Thomas, published on Healing Water website, 2009.
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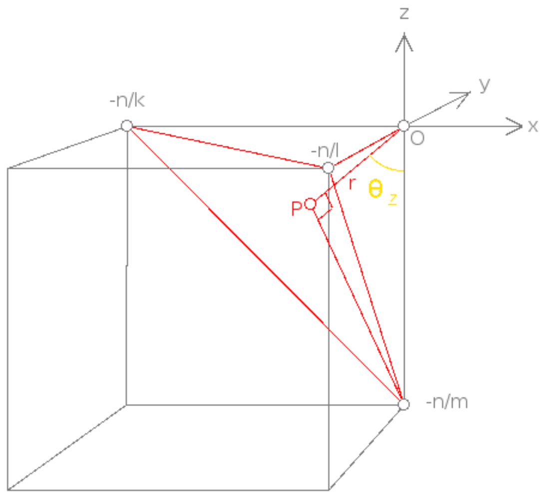
Dornach, Christmas 2001 (the article is in English).

Appendix A

It is shown below that the homogeneous coordinates of a plane are proportional to the components of a vector normal to that plane.



A plane {k l m n}, say, is shown as a red triangle intersecting the coordinate axes at -n/k, -n/l and -n/m. The red line OP through the origin O is perpendicular to the plane and of of length r, with direction cosines $\cos\theta_x$ $\cos\theta_y$ and $\cos\theta_z$ (only θ_z is shown). Now

$$\cos(\theta_z) = -\frac{rm}{n}$$

and similarly

$$\cos\left(\theta_y\right) = -\frac{rl}{n}$$

$$\cos\left(\theta_x\right) = -\frac{rk}{n}$$

Hence $\cos\theta_x : \cos\theta_y : \cos\theta_z = k : l : m$.

If the coordinates of P are {x y z 1} then

$$\cos\theta_x : \cos\theta_y : \cos\theta_z = x/r : y/r : z/r$$

so that

$$k : l : m = x : y : z$$

i.e. the first three plane coordinates are in the same ratio as the components of the normal to the plane.

Appendix B

The use of homogeneous coordinates for general tetrahedra of reference will be explained here. Consider a real tetrahedron **T** represented by the square 4x4 matrix t_{jk}, where the columns are the Cartesian homogeneous coordinates of the vertices, all composed of real numbers. In coordinates referred to **T** as reference tetrahedron, the vertices of **T** have the coordinates {1 0 0 0}, {0 1 0 0}, {0 0 1 0} and {0 0 0 1}. If the vertex V of **T** is taken to be represented by v_j = {1 0 0 0} then $w_j=t_{jk}v_k$ gives the Cartesian coordinates w_j of V, as w_j equals the first column of t_{jk}, and similarly for the other vertices of **T** (note that the convention is adopted that repeated suffices imply summation, so $t_{jk}v_k$ means $t_{j0}v_0+t_{j1}v_1+t_{j2}v_2+t_{j3}v_3$). Thus t_{jk} transforms from coordinates referred to **T** to Cartesian coordinates. Clearly $v_j=t_{jk}^{-1}w_k$ performs the reverse operation, so t_{jk}^{-1} - the inverse matrix of t_{jk} - transforms Cartesian coordinates into coordinates referred to **T**. This applies to points and dually to planes. The well known relation of the transformation of planes to that of points is derived as follows. Using the notation $\mathbf{u}^\mathbf{T}$ for the transpose of the vector **u** (or a matrix), if **x=Tx'** transforms points from projective to Cartesian coordinates as above, then if **x** lies in the plane **u**, $\mathbf{u}^\mathbf{T}\mathbf{x}=0$ i.e. $\mathbf{u}^\mathbf{T}\mathbf{Tx'}=0$, and since $\mathbf{u'}^\mathbf{T}\mathbf{x'}=0$ it follows that $\mathbf{u'}^\mathbf{T}=\mathbf{u}^\mathbf{T}\mathbf{T}$ giving $\mathbf{u'}=\mathbf{T}^\mathbf{T}\mathbf{u}$ for transforming a plane to projective coordinates.

Cartesian coordinates are referred to the tetrahedron

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where the origin is at {0 0 0 1} and e.g. {1 0 0 0} is the point at infinity on the x-axis.

A path-curve transformation is represented by the matrix

$$\begin{bmatrix} r\cos\theta & 0 & r\sin\theta & 0 \\ 0 & s & 0 & 0 \\ -r\sin\theta & 0 & r\cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

To apply (11) the canonical (i.e. diagonal) form of this transformation is required, which is

$$\begin{bmatrix} r\,e^{i\theta} & & & \\ & s & & \\ & & r\,e^{-i\theta} & \\ & & & 1 \end{bmatrix} \tag{B1}$$

These coordinates are no longer all real and so must refer to another type of invariant tetrahedron with two imaginary vertices (it is customary to refer to points with complex numbers for their coordinates as "imaginary" rather than "complex" points). If two planes of a real tetrahedron are parallel and horizontal, with one of its edges vertical meeting the horizontal planes in two real vertices, then the vertices on the line at infinity where the horizontal planes meet are converted into the absolute circling points on that line. That gives a tetrahedron within which a transform such as (B1) may be applied. The Cartesian coordinates of the resulting semi-imaginary tetrahedron are:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & h & 0 & 0 \\ i & 0 & -i & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \tag{B2}$$

where {0 h 0 1} and {0 0 0 1} are the two real invariant points on the y-axis, one being the Cartesian origin. The x- and z-axes are taken to be horizontal and the y-axis vertical so that the first and third vertices are complex conjugate vectors related to the absolute circling points at infinity on the two real invariant planes. This is also the transformation matrix **T** to Cartesian coordinates from coordinates referred to this as tetrahedron of reference, as explained in the first paragraph. Its inverse transforms Cartesian points into points referred to **T**. Transformations such as (B1) may now be referred to this tetrahedron. Thus to apply (11) a Cartesian plane P is transformed into coordinates referred to **T** by multiplying by the transpose of (B2), then a transformation such as (B1) is applied to the plane as in (11), and finally the resulting pivot point is transformed back to Cartesian form by multiplying by (B2). Hence there is no need to evaluate the inverse of (B2) in this case.

For a "fully imaginary" tetrahedron for transformations with all coordinates imaginary, the matrix is

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ i & 0 & -i & 0 \\ 0 & i & 0 & -i \end{bmatrix} \tag{B3}$$

where the second and fourth columns are also complex comjugate, representing the absolute circling points on the other pair of now imaginary invariant planes. Note also that the final Cartesian coordinates after the transformation may well all appear to be imaginary, but dividing all coordinates by the fourth one converts to real Cartesian coordinates, as a common imaginary or complex common factor may be removed from homogeneous coordinates.

Appendix C

When a semi- or fully-imaginary tetrahedron of reference is used it is useful to find out how the coordinates of planes through a fixed point on the y-axis and the y-coordinates of their pivot points are related. The calculations below need not be carried out explicitly as shown, but using complex number routines in a computer the procedure in Appendix B may be carried out directly. The purpose of this Appendix is to demonstrate the theory behind such calculations.

A plane through Q{0 q 0 1} is $\{u_0\ u_1\ u_2\ {-qu_1}\}$, say, in homogeneous form, and this is converted to the semi-imaginary-tetrahedron coordinates by multiplying by the transpose of (B2), giving

$$\begin{bmatrix} u_0 + i u_2 \\ u_1 (h - q) \\ u_0 - i u_2 \\ -q u_1 \end{bmatrix} \qquad \text{(C1)}$$

Applying (11) to this and multiplying by (B2):

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & h & 0 & 0 \\ i & 0 & -i & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dfrac{X}{u_0+i u_2} \\ \dfrac{Y}{u_1(h-q)} \\ \dfrac{Z}{u_0-i u_2} \\ -\dfrac{W}{q u_1} \end{bmatrix} = \begin{bmatrix} \dfrac{X}{u_0+i u_2}+\dfrac{Z}{u_0-i u_2} \\ \dfrac{hY}{u_1(h-q)} \\ \dfrac{iX}{u_0+i u_2}-\dfrac{iZ}{u_0-i u_2} \\ \dfrac{Y}{u_1(h-q)}-\dfrac{W}{q u_1} \end{bmatrix} \tag{C2}$$

so the y-coordinate of the pivot point in Cartesian coordinates is

$$\frac{\left\{\dfrac{hY}{u_1(h-q)}\right\}}{\left\{\dfrac{Y}{u_1(h-q)}-\dfrac{W}{q u_1}\right\}} = \frac{q h Y}{-W h+q(Y+W)} \tag{C3}$$

which is of the same general form in q as that given by Edwards (1993), where we use q for his h'. Note that in this case X and Z are complex conjugate numbers, so the final Cartesian x- and z-coordinates in (C2) are real. Apart from q, (C3) is independent of the other coordinates of the original plane through Q{0 q 0 1}, showing that all planes containing Q have their pivot points in a plane with fixed y-coordinate given by (C3). q=∞ for planes intersecting the y-axis at infinity, giving the y-coordinate of the corresponding pivot point as

$$\frac{hY}{Y+W} \tag{C4}$$

Thus if the original surface possesses a tangent plane **u** parallel to the y-axis then the pivot-transform contains a point at infinity on a horizontal line in **u** at this height, as the Cartesian x- and z-coordinates are infinite in (C2).

The pivot-point in (C2) lies on the y-axis if Cartesian x=z=0 i.e. q=0 or q=h or u_1=0. In the first two cases the pivot-transform surface contains the points {0 0 0 1} or {0 h 0 1} respectively. The third case is singular as then the tangent plane is U{u_0 0 u_2 0} i.e. it contains the y-axis and so q is indeterminate in (C3). It will depend upon the surface being transformed i.e. upon q for tangent planes of the surface infinitely close to U. For example, for a vortex with its axis of symmetry coinciding with the y-axis and containing the point at infinity on the y-axis (a so-called "watery vortex"), q=∞ so its pivot transform crosses the y-axis at the height given by (C4).

For a "fully imaginary" tetrahedron with only two real lines, we multiply (C1) by the transpose of (B3):

$$\begin{bmatrix} u_0+iu_2 \\ u_1(1-iq) \\ u_0-iu_2 \\ u_1(1+iq) \end{bmatrix}$$

and applying (11) and multiplying by (B3):

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ i & 0 & -i & 0 \\ 0 & i & 0 & -i \end{bmatrix} \begin{bmatrix} \dfrac{X}{u_0+iu_2} \\ \dfrac{Y}{u_1(1-iq)} \\ \dfrac{Z}{u_0-iu_2} \\ \dfrac{W}{u_1(1+iq)} \end{bmatrix} = \begin{bmatrix} \dfrac{X}{u_0+iu_2}+\dfrac{Z}{u_0-iu_2} \\ \dfrac{Y}{u_1(1-iq)}+\dfrac{W}{u_1(1+iq)} \\ \dfrac{iX}{u_0+iu_2}-\dfrac{iZ}{u_0-iu_2} \\ \dfrac{iY}{u_1(1-iq)}-\dfrac{iW}{u_1(1+iq)} \end{bmatrix}$$

The Cartesian y-coordinate is, taking Y=r+i.s and W=r-i.s :

$$\frac{(r+is)(1+iq)+(r-is)(1-iq)}{i(r+is)(1+iq)-i(r-is)(1-iq)}=\frac{sq-r}{rq+s} \tag{C5}$$

so again if one real edge is the vertical Cartesian y-axis then planes through a fixed point on the y-axis have their pivot points in a plane with a fixed y-coordinate. q=∞ for planes intersecting the y-axis at infinity, when the y-coordinate of the pivot points is

$$\frac{s}{r} \tag{C6}$$

The x- and z-coordinates are infinite and the situation is as for the semi-imaginary case. Again planes through he Cartesian origin with q=0 contain that point as their pivot points. The values of r and s are derived from (10), noting that a and c are complex conjugate numbers, as are b and d in this case. However the use of complex number routines in a computer program enables X Y Z and W to be found directly without recourse to this theory.