Pivot Transforms and Legendre Transforms

Legendre Transforms are a method of analysing a shape into "harmonics" analogous to Fourier Transforms for time waveforms. In the radial case a circle (or sphere in three dimensions) is the "fundamental" and petal-like forms are the shapes of the higher "harmonics". Legendre discovered a set of orthogonal functions known as Legendre polynomials which enable the analysis of a function as follows:

$$f(\mu)=\sum_{r=0}^{\infty} c_r P_r(\mu) \tag{1}$$

where the $P_r(\mu)$ are the Legendre polynomials and c_r are their amplitudes.

The author applied this to a projective vortex and noticed that the harmonic envelope resembled an egg form. After trying many vortices with different parameters it became clear that an egg is not the most general form, but rather that all envelopes inspected resembled pivot transforms of various kinds. It was then necessary to see whether those envelopes could be fitted to actual pivot transforms.

The pivot transform was discovered by Lawrence Edwards and is described in his book *The Vortex of Life*.

The process is summarised in the following diagram:

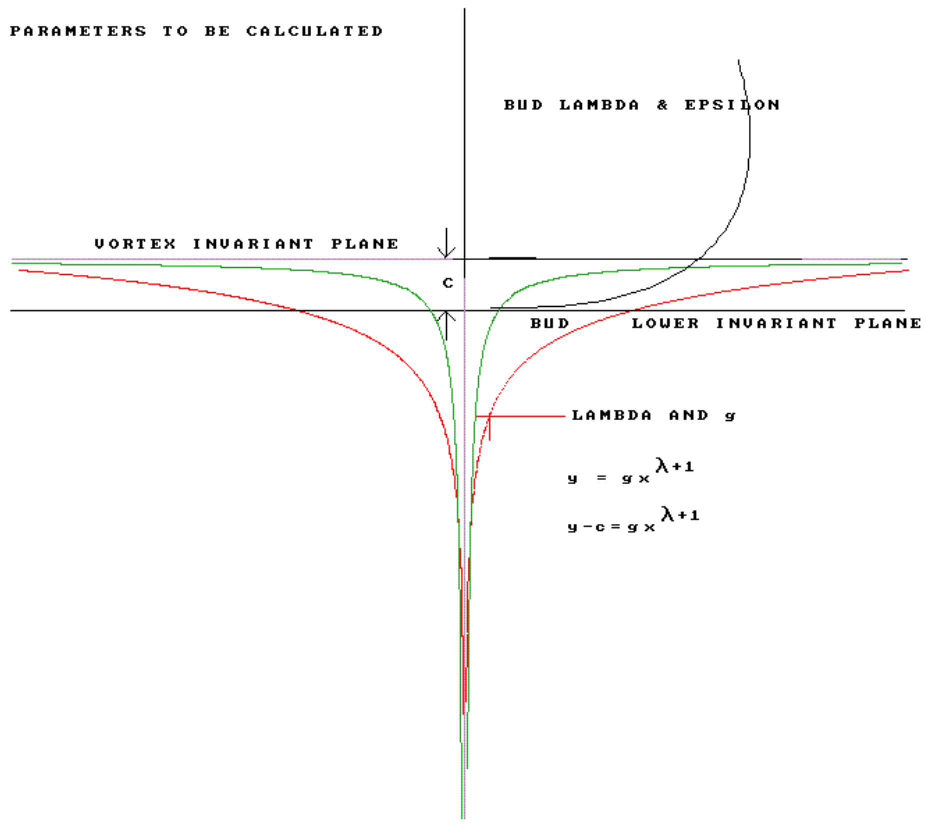

Figure 1

A linear projective transformation characterised by its λ and ϵ parameters has its lower invariant plane shown in the diagram as the "bud lower invariant plane". The upper invariant plane is above the diagram. The watery vortex to be transformed is shown with its upper invariant plane a distance *c* above the bud plane. Its own distinct λ parameter and size parameter *g* control its shape. Then given a harmonic envelope from a Legendre Transform of a vortex V_1, the problem to be solved is to find the parameters in the above diagram such that the pivot transform of a second vortex V_2 fits that envelope.

The author wrote a computer program to find the best fit. Generally this showed that the harmonic envelopes can indeed be fitted to pivot transforms, and in particular that the reverse transforms of pivot forms, treated as harmonic envelopes, in all cases produced vortices. Typical results are shown below:

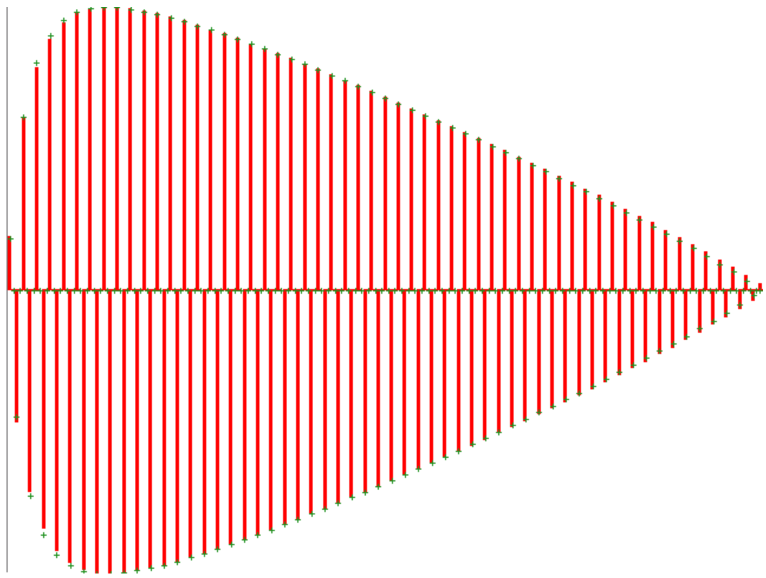

Figure 2

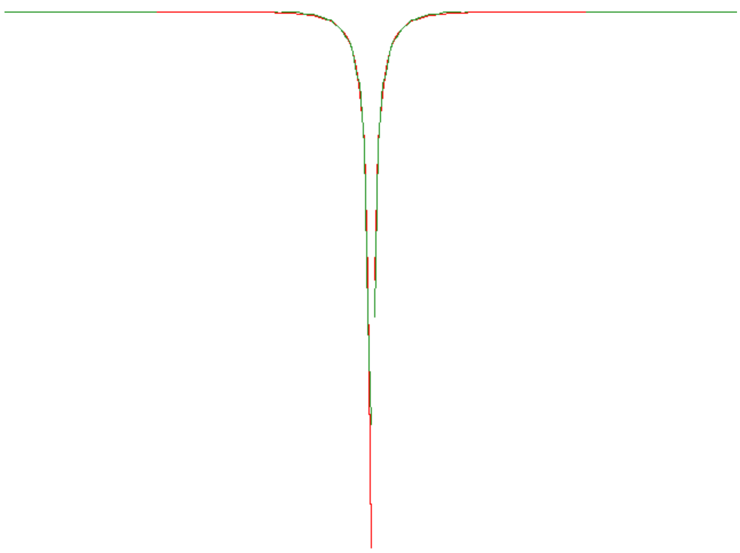

Figure 3

Figure 2 shows in red the Legendre harmonic envelope for a vortex V_1 with $\lambda = -2.9$, while the green crosses show the fitted pivot transform. Taking the c_r from the pivot form and calculating the vortex shape using equation (1) yielded a vortex form virtually identical with that originally transformed, as shown in Figure 3, where the green form is the reverse-engineered one and the red one (mostly obscured) was the original.

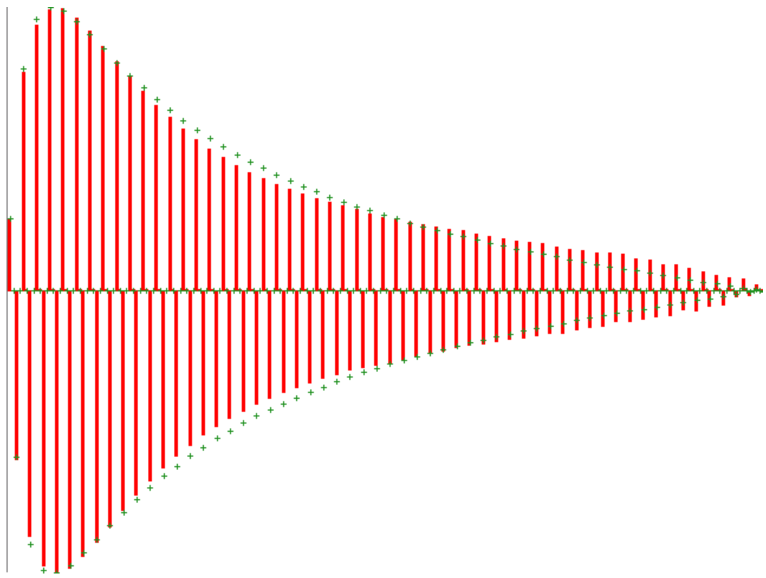

Figure 4

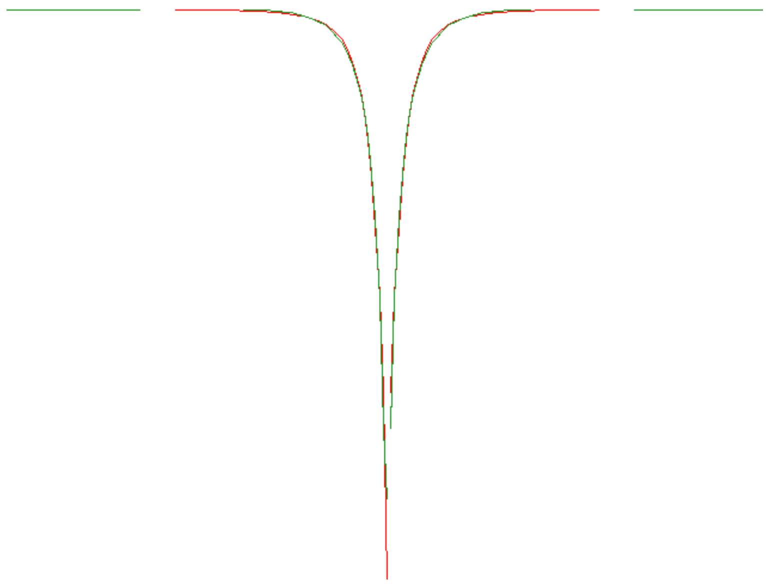

Figure 5

Figures 4 and 5 are similar for an original vortex V_1 with $\lambda = -3.5$.

Figures 6 and 7 below show a case for $\lambda = -2.5$ where the original harmonic envelope is not smooth. This is because the lower λ value results in greater discontinuities at the x-coordinate boundaries -1 and 1 (required for the Legendre Transform) as the curve has not "levelled off" at the those cut-off points.

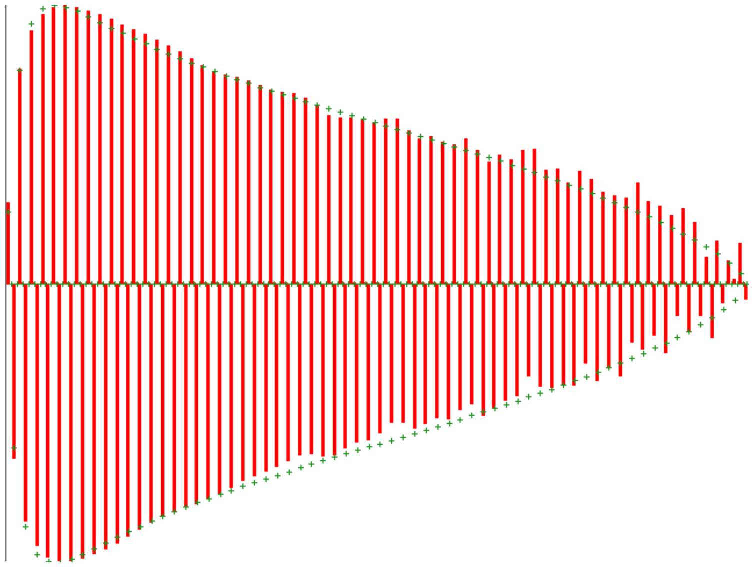

Figure 6

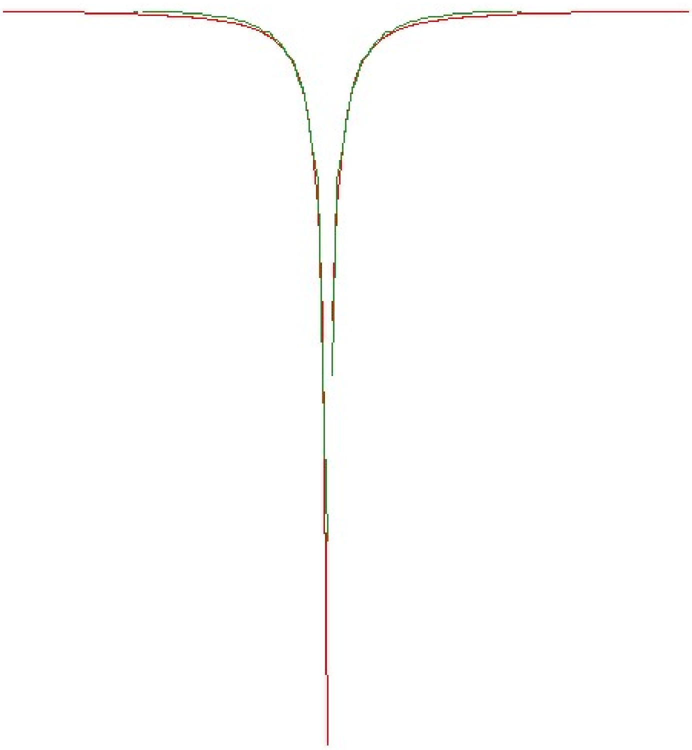

Figure 7

Many other examples were shown in the lecture, and the conclusions to be drawn are:

- Pivot transforms of projective vortices of all kinds, when treated as Legendre harmonic envelopes, are Legendre Transforms of vortices i.e. using Equation (1) vortices are always obtained.
- Legendre Transforms of projective vortices have harmonic envelopes which may be fitted to pivot transforms, generally very well except in extreme cases.
- It cannot be proved analytically that such envelopes "are" pivot transforms as the two transforms are quite different in kind, but the empirical results of comparing them are perhaps surprising and interesting.

The motive behind this admittedly empirical investigation is as follows. Legendre Transforms are fundamental to the way wave-functions are derived in quantum mechanics, for they and their derivatives provide solutions to Legendre's differential equation which arises e.g. in the description of the motion of an electron on a sphere (used to analyse the behaviour of the hydrogen atom). The surface spherical harmonics satisfying Laplace's equation $\nabla^2 \phi = 0$ on the surface of a sphere - which guarantees solutions that are continuous, single-valued and finite - involve Legendre's equation.

On the other hand the pivot-transform concerns relationships between space and counterspace at a macroscopic level. A relationship between these two transforms, empirical or otherwise, is being investigated as a practical way of approaching quantum phenomena from a counterspace perspective.

Nick Thomas, May 2007