INFINITESIMAL TRANSFORMS

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Given a linear transform **x' = Ax** we can interpret it as a transformation giving path curves. The question is: how to reduce it to infinitesimal form, for as it stands the steps are relatively large. Starting in the plane so that **A** is a 3x3 matrix, we take the invariant triangle to be the triangle of reference for the homogeneous coordinates, in which case **A** is a diagonal matrix with the characteristic roots in the leading diagonal, assuming a collineation in classical canonical form:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} a & & \\ & b & \\ & & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{1}$$

Initially we will assume that a,b and c are positive and distinct. Clearly we can reduce the step size by taking roots of a, b and c e.g. if we take the square root of each then we obtain the same path curves but with twice the number of steps to go from a given initial to a given final point. Generally we take n'th roots with n as large as we please to reduce the step size, so as n → ∞ the transformation becomes infinitesimal. Thus

$$\frac{x'}{x} = a^{1/n}$$

$$\frac{\log(a)}{n} = \log\left(\frac{x'}{x}\right)$$

$$= \log\left(\frac{x+dx}{x}\right) = \log\left(1+\frac{dx}{x}\right)$$

For large n, δx becomes very small, so log{1+δx/x} → δx/x, giving

$$\frac{\delta x}{x} = \frac{\log a}{n} \tag{2}$$

Similarly for δy/y and δz/z. Taking ratios we have

$$\frac{\delta y}{\delta x} = \frac{y \log b}{x \log a}$$

and as n → ∞ this is

$$\frac{\partial y}{\partial x} = \frac{y \log b}{x \log a} \tag{3}$$

We get similar equations for ∂y/∂z and ∂z/∂x.
Integrating we get

$$\frac{1}{\log(b)}\int\frac{\partial y}{y}=\frac{1}{\log(a)}\int\frac{\partial x}{x}$$

$$\frac{\log(y)}{\log(b)}=\frac{\log(x)}{\log(a)}+\log(m_1)$$

where m_1 is an arbitrary constant, so

$y^{\beta} = m_1\, x^{\alpha}$ where $\beta = 1/\log(b)$ and $\alpha = 1/\log(a)$

Similarly

$x^{\alpha} = m_2\, z^{\gamma}$ where $\gamma = 1/\log(c)$

and

$z^{\gamma} = m_3\, y^{\beta}$ where $m_1\, m_2\, m_3 = 1$

Thus $z^{2\gamma} = m_3^2\, y^{\beta}\, (m_1.x^{\alpha})$

and so

$x^{\alpha}\, y^{\beta}\, z^{-2\gamma} = k$ where $k = m_1/m_3^2$

This equation is satisfied by the collineation, which may be verified as follows. Assume it is satisified for some initial point {x,y,z}, giving k. After being transformed, the new point is {ax,by,cz}. Thus to satisfy the above equation we require

$a^{\alpha} b^{\beta}\, c^{-2\gamma} = 1$

so

$\alpha\, \log(a)+\beta\, \log(b) = 2\, \gamma\, \log(c)$

and substituting for α, β and γ we find this is an identity, which verifies the equation. But it is not a homogeneous equation unless $\alpha+\beta-2\gamma=0$, because multiplying all the homogeneous coordinates by a common factor must leave the equation unchanged.

A more general approach, useful later on, is to see that

$x^{p\alpha}\, y^{q\beta}\, z^{r\gamma} = k$ satisfies the collineation if

$a^{p\alpha}\, b^{q\beta}\, c^{r\gamma} = 1$, so we require $p+q+r = 0$ (as $0 = p\alpha\log(a)+q\beta\log(b)+r\gamma\log(c) = p+q+r$).

To obtain path curves we must also satisfy the homogeneous condition, so we have

$p\alpha+q\beta+r\gamma= 0$ as well as $p+q+r = 0$ (4)

We may choose p arbitrarily and solve for q and r, giving

$q = p(\alpha)/(\gamma-\beta)$ and $r = p(\alpha-\beta)/(\beta-\gamma)$ and thus

$$x^{p\alpha}\, y^{p\beta(\alpha-\gamma)}\, z^{p\gamma(\beta-\alpha)} = k \tag{5}$$

so the general equation for a path curve is

$$x^u\, y^v\, z^w = k \tag{6}$$

for some constant k, where $u+v+w = 0$. This is related to the collineation by

$$u = p\alpha(\gamma-\beta),\ v = p\beta(\alpha-\gamma),\ w = p\gamma(\beta-\alpha) \tag{7}$$

where p is an arbitrary constant and

$$\alpha = 1/\log(a),\ \beta = 1/\log(b),\ \gamma = 1/\log(c).$$

Curves described by (6) are the simplest possible without inflexions or loops (as indicated by the equations like (3))

<u>Complex Conjugate Characteristic Roots</u>

Equation (6) still applies when two of the characteristic roots in (1) are complex conjugate, say a and b which are then of the form $re^{i\phi}$, $re^{-i\phi}$. In that case the initially real transformation in the reference coordinate system is

$$\begin{bmatrix} r\cos\theta & r\sin\theta & \\ -r\sin\theta & r\cos\theta & \\ & & c \end{bmatrix}$$

and the characteristic vectors are { 1 : ± i : 0 } giving the transformation matrix from the canonical to the reference domain as

$$\begin{bmatrix} 1 & 1 & \\ i & -i & \\ & & 1 \end{bmatrix} \tag{8}$$

Its inverse gives the transformation from the reference coordinates to the canonical ones:

$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{2}i & \\ \frac{1}{2} & \frac{1}{2}i & \\ & & 1 \end{bmatrix} \tag{9}$$

so applying this to a point {x,y,z} we obtain

$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{2}i & \\ \frac{1}{2} & \frac{1}{2}i & \\ & & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(x-iy) \\ \frac{1}{2}(x+iy) \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{2}re^{-i\phi} \\ \frac{1}{2}re^{i\phi} \\ z \end{bmatrix} \tag{10}$$

so that the canonical coordinates are complex numbers.

The equation (6) for a path curve is now expressed with bars above the coordinates to indicate that they are complex numbers in the canonical reference system:

$$\bar{x}^u \bar{y}^v \bar{z}^w = k \tag{11}$$

We recall (c.f. (7)) that

$u = p\alpha(\gamma-\beta)$; $v = p\beta(\alpha-\gamma)$; $w = -u-v$

where

$\alpha = 1/\log(a) = 1/\log(e^{i\theta})$ in this case, $= -i/\theta$

$\beta = 1/\log(b) = i/\theta$

$\gamma = 1/\log(c)$

so (11) becomes, using (10),

$$\left(\frac{1}{2}re^{-i\phi}\right)^{-\frac{ip}{\theta}\left(\frac{1}{\log(c)}-\frac{i}{\theta}\right)} \left(\frac{1}{2}re^{i\phi}\right)^{\frac{ip}{\theta}\left(\frac{-i}{\theta}-\frac{1}{\log(c)}\right)} z^{-u-v} = k$$

which we re-arrange as

$$\left[\left(\frac{r}{2z}\right)^{\frac{-2ip}{\theta\log(c)}}\right] e^{\frac{2ip\phi}{\theta^2}} = k$$

Assuming k is imaginary we end up with a real equation (re-defining k)

$$\left(\frac{r}{2z}\right)^{\frac{-2p}{\theta\log(c)}} e^{\frac{2p\phi}{\theta^2}} = k$$

which we re-arrange as

$$\left(\frac{r}{z}\right)^{\frac{2p}{\theta\log(c)}} = \left(k2^{\frac{2p}{\theta\log(c)}}\right) e^{\frac{2p\phi}{\theta^2}}$$

Raising both sides to the power $\theta\log(c)/2p$ (which is constant) we obtain

$$(r/z) = (r_0/z_0)e^{g\phi}$$

which is a logarithmic spiral in the Cartesian case, where $r_0/z_0 = 2k^{\theta\log(c)/2p}$ and $g = \log(c)/\theta$. We may revert to Cartesian coordinates by setting $z = z_0 = 1$.

Negative Characteristic Roots

a, b or c may be negative in (1). Since the coordinates are homogeneous we may always reduce this to one negative value. The result is that the coordinate corresponding to the negative eigenvalue oscillates in sign while the other two do not. The derivation of (6) then involves the log of a negative number in the indices i.e. a complex number.

Starting at any point on the curve, the discreet transformation (1) follows an oscillating set of points. If we apply (1) twice with c<0, say, then we get the point

$$\{a^2x, b^2y, c^2z\} \tag{12}$$

as if the transformation were all positive. This transformation is a simple one as described by (6), and from any starting point the oscillating points jump between two path curves of type (6), each starting point selecting a different pair as illustrated below:

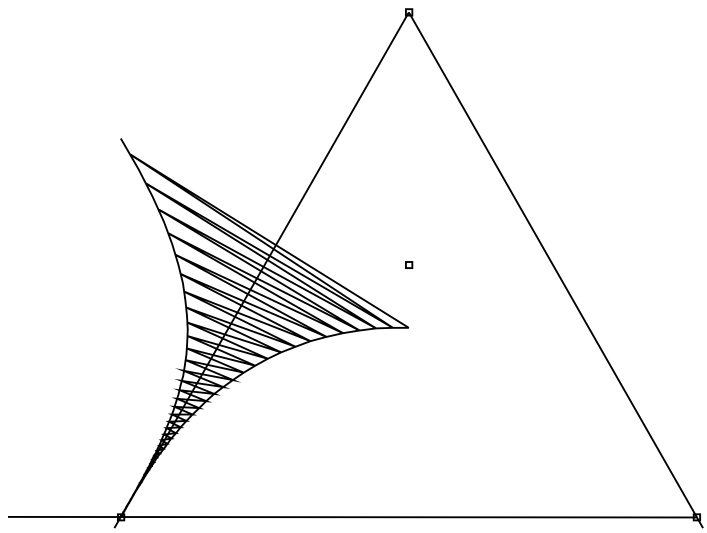

Figure 1

From (7) we have $u = p\alpha(\gamma-\beta)$, $v = p\beta(\alpha-\gamma)$, $w = p\gamma(\beta-\alpha)$ where

$$\gamma = \frac{1}{\log(-c)} = \frac{1}{\log(c)+i\pi} = \frac{\log(c)-i\pi}{(\log(c))^2+\pi^2} \tag{13}$$

so (5) becomes

$$k = x^{p\alpha\gamma}\, x^{-p\alpha\beta}\, y^{p\alpha\beta}\, y^{-p\beta\gamma}\, z^{p\beta\gamma}\, z^{-p\alpha\gamma} = (y/x)^{p\alpha\beta}\{x^{p\alpha\gamma}\, y^{-p\beta\gamma}\, z^{p\gamma(\beta-\alpha)}\}$$

and

$$k^{\gamma'} = (y/x)^{p\alpha\beta\gamma'}\{x^{p\alpha\gamma\gamma'}\, y^{-p\beta\gamma\gamma'}\, z^{p\gamma\gamma'(\beta-\alpha)}\}$$

where γ' is the complex conjugate of γ, & thus

$$x^{p\alpha\gamma\gamma'}\ y^{-p\beta\gamma\gamma'}\ z^{p\gamma\gamma'(\beta-\alpha)} = k^{\gamma'}\ (x/y)^{p\alpha\beta\gamma'}$$

From (13) $\gamma\gamma' = 1/\{(\log(c))^2+\pi^2\}$ so we may remove the common exponent $1/\{(\log(c))^2+\pi^2\}$ to give

$$x^{p\alpha}\ y^{-p\beta}\ z^{p(\beta-\alpha)} = k^{\log(c)+i\pi}\ (x/y)^{p\alpha\beta\{\log(c)+i\pi\}} \qquad (14)$$

The LHS is a simple path curve and for a given x/y we only obtain a real result if p is suitably chosen i.e. for a suitable power of the original transformation. The imaginary part is

$$k^{i\pi}\ (x/y)^{i\pi p\alpha\beta} = e^{i\pi[p\alpha\beta\log(x/y)+\log(k)]} \qquad \text{so that } p\alpha\beta\ \log(x/y)+\log(k) = 2n \qquad (15)$$

for some integer n.

Rearranging (14), noting that $k^{i\pi}(x/y)^{i\pi p\alpha\beta} = 1$ now, we get

$$x^{\alpha[1-\beta\log(c)]}\ y^{\beta[\alpha\log\{c)-1]}\ z^{(\beta-\alpha)} = k^{\log(c)/p} \quad = k^{\log(c)[\alpha\beta\{\log(x)-\log(y)\}]/[2n-\log(k)]}$$
$$= x^{\alpha\beta\log(k)\log(c)/[2n-\log(k)]}\ y^{-\alpha\beta\log(k)\log(c)/[2n-\log(k)]}$$

or

$$x^{-\alpha\{2n[1-\beta\log(c)]-\log(k)\}}\ y^{\beta\{\log(k)-2n[1-\alpha\log(c)]\}}\ z^{(\beta-\alpha)[2n-\log(k)]} = 1 \qquad (16)$$

This apparently gives one set of simple path curves for each choice of n. However, we may set c = 1 and rescale α and β accordingly in which case we get the equation in its simplest form:

$$x^{\alpha}\ y^{-\beta}\ z^{\beta-\alpha} = 1 \qquad (17)$$

There are just two simple path curves given by this depending on the signs of $\alpha$and β. For example if α is positive we may raise the whole equation to the power $2/\alpha$ to give $x^2y^{2\beta/\alpha}z^{2(\beta-\alpha)/\alpha} = 1$, and then choosing positive values for y and z we obtain two solutions for x of equal and opposite sign, giving the two simple path curves in Figure 1.

We conclude that a negative eigen value gives rise to a pair of simple path curves.

It is interesting to digress for a moment at this point and consider correlations. The square matrix in (1) may be regarded as a correlation in which case {x',y',z'} represents a line. The square of the correlation gives a collineation either of points or lines and hence we obtain two path curves, one described by points and the other enveloped by lines. This is illustrated below:

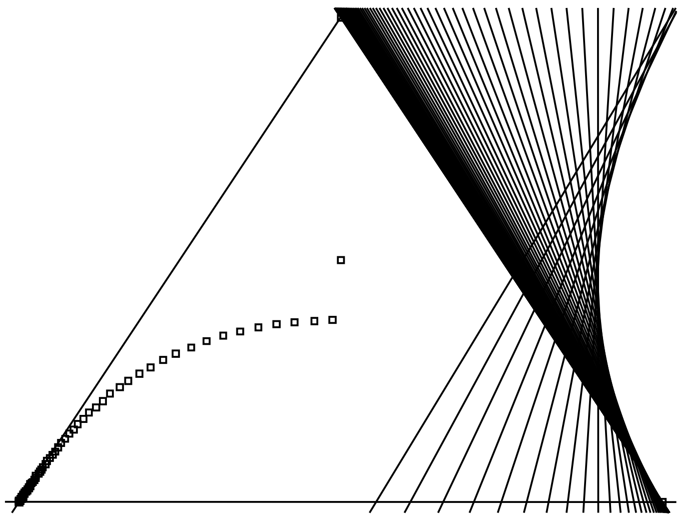

Figure 2

Since a line with the same coordinates as a point is the polar of that point, the envelope is the polar of the second simple path curve in Figure 1.

Repeated Characteristic Roots

If two of the characteristic roots are equal it follows that the ratio of the corresponding coordinates is constant, and putting $\alpha = \beta$ in (5) we obtain a straight line given by

$$xy^{-1} = k \tag{18}$$

If all three are equal we obtain the identity transformation.

Three-Dimensional Path Curves

Again assuming initially a simple classical canonical form for the transformation, which is a diagonal matrix, and again with distinct positive characteristic roots, we have in place of (1)

$$\begin{bmatrix} x' \\ y' \\ z' \\ t' \end{bmatrix} = \begin{bmatrix} a & & & \\ & b & & \\ & & c & \\ & & & d \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} \tag{19}$$

We obtain equations similar to (2) and clearly we end up with an equation equivalent to (6)

$$x^u\, y^v\, z^w\, t^s = k \tag{20}$$

where $\quad u+v+w+s = 0$

and $\quad u = p_1\alpha,\;\; v = p_2\beta,\; w = p_3\gamma,\; s = p_4\mu$ with $\mu=1/\log(d)$ and $\sum p_i = 0$ (c.f. (4)).

Given (19) we know $\alpha\ \beta\ \gamma\ \mu$, so to determine (20) we must find the p_i. As we have the two equations

$$\begin{aligned} &\sum p_i = 0 \\ \text{and}\quad & \\ &\alpha p_1+\beta p_2+\gamma p_3+\mu p_4 = 0 \end{aligned} \tag{21}$$

we are free to choose two of the p_i and solve equations (21) to find the other two.

(20) is not the equation of a curve, but a surface – a path-surface. The fact that we may choose two of the p_i freely shows that there are distinct types of such a surface for a given transformation, the parameter k in (20) selecting one of a given type. Every point of space lies in such a surface as substituting {x,y,z,t} in the LHS of (20) yields a number which determines k. Two surfaces of the same type do not intersect as a given point determines k uniquely. Thus a set of path-surfaces of a given type completely covers space without overlap. Every point in such a surface clearly transforms into another point lying in it by definition, so if two distinct surfaces (of different type) intersect in a curve every point of the curve must transform into another point also on the curve i.e. we have a path curve. Thus in three dimensions a path curve may be described by two equations like (20).

If two of the characteristic roots in (19) are complex conjugate, then if we hold the ratio z/t relating to the real characteristic roots constant we define a plane μ in which the two-dimensional analysis applies i.e. we obtain a logarithmic spiral in μ. As μ varies with z/t a spiral path-surface arises made up of the individual spirals. In this case two of the canonical coordinates are complex conjugate and two are real. If instead we hold x/y constant to give a plane μ for the characteristic roots $re^{\pm i\theta}$, then we must raise the collineation to a suitable power σ so that σθ=nπ. The result is a set of simple path curves and the surface meets μ in a discrete set of them determined by increasing n.

If the other two characteristic roots are also complex conjugate then we obtain a path curve directly which is a twisted spiral of varying radius as will be shown below. A more detailed analysis, redundant here, shows that there is only one type of path-surface which is why path curves arise directly.

Repeated Characteristic Roots

If in (19) c=d, say, then z/t is constant and the path curve lies in a plane. Its form depends upon the other two characteristic roots and may be simple, spiral or paired. Two pairs of equal characteristic roots yield two planes and so the path curves are straight lines. If three characteristic roots are equal then the planar path curve has two equal characteristic roots and is thus also a straight line. Four equal characteristic roots yield, of course, the identity transformation.

Parametric Equations for a Path Curve

It is possible to obtain parametric equations for a path curve. Since it is the intersection of two path-surfaces of distinct types, if we have two equations like (20) for these surfaces, say $x^a y^b z^c t^d = k$ and $x^p y^q z^r t^s = m$, then noting that a+b+c = -d and p+q+r = -s we have the equation

$$\begin{bmatrix} a & b \\ p & q \end{bmatrix} \begin{bmatrix} \log\left(\frac{x}{t}\right) \\ \log\left(\frac{y}{t}\right) \end{bmatrix} = \begin{bmatrix} 1 & -c \\ h & -r \end{bmatrix} \begin{bmatrix} \log k \\ \log\left(\frac{z}{t}\right) \end{bmatrix}$$

where $m=k^h$ for suitable h, so that

$$\begin{bmatrix} \log\left(\frac{x}{t}\right) \\ \log\left(\frac{y}{t}\right) \end{bmatrix} = \begin{bmatrix} a & b \\ p & q \end{bmatrix}^{-1} \begin{bmatrix} 1 & -c \\ h & -r \end{bmatrix} \begin{bmatrix} \log k \\ \log\left(\frac{z}{t}\right) \end{bmatrix} = \begin{bmatrix} \frac{q-bh}{aq-pb} & \frac{br-qc}{aq-pb} \\ \frac{ah-p}{aq-pb} & \frac{pc-ar}{aq-pb} \end{bmatrix} \begin{bmatrix} \log k \\ \log\left(\frac{z}{t}\right) \end{bmatrix}$$

and hence

$$(x/t) = k^{(q-bh)/(aq-pb)} (z/t)^{(br-qc)/(aq-pb)} \ ; \ (y/t) = k^{(ah-p)/(aq-pb)} (z/t)^{(pc-ar)/(aq-pb)}$$

which shows that in Cartesian coordinates we have parametric equations of the form

$$x = m_1 z^{\alpha} \ ; \quad y = m_2 z^{\beta} \quad \text{for a path curve}$$

where m_1,m_2 are freely selectable (instead of k and m) to determine the path-surfaces.

This suggests the following homogeneous parametric equations with t as parameter:

$$x = k_1 t^{\alpha} \;;\; y = k_2 t^{\beta} \;;\; z = k_3 t^{\gamma} \tag{22}$$

so that $(x/t) = k_1 t^{\alpha-1} = k_1 k_3^{(\gamma-1)/(\alpha-1)}(z/t)^{(\alpha-1)/(\gamma-1)}$

and similarly $(y/t) = k_2 k_3^{(\gamma-1)/(\beta-1)}(z/t)^{(\beta-1)/(\gamma-1)}$

Hence $(br-qc)/(aq-pb) = (\alpha-1)/(\gamma-1)$ and $(pc-ar)/(aq-pb) = (\beta-1)/(\gamma-1)$;

$k_1 = m_1 m_2^{(\gamma-1)/(\alpha-1)}$ and $k_2 = m_1 m_2^{(\gamma-1)/(\beta-1)}$.

Clearly given α β γ we can find infinitely many solutions for a b c and p q r for the infinitely many pairs of path-surfaces intersecting in pairs in the path curve given by (22).

t is a valid parameter (i.e. it varies continuously) as the path curve lies wholly in one region of the invariant tetrahedron, and we may select the signs of the homogeneous coordinates to make t always positive in that region (which is necessary as α β γ are generally real numbers).

We may associate a discrete collineation with (22) by supposing that d in (19) is given, so that if t transforms into t_1 then $t_1/t = d$. Thus $x_1/x = (t_1/t)^{\alpha} = d^{\alpha}$, and similarly for y and z, giving

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ t_1 \end{bmatrix} = \begin{bmatrix} d^{\alpha} & & & \\ & d^{\beta} & & \\ & & d^{\gamma} & \\ & & & d \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} \tag{23}$$

This suggests a simple way of deriving (22) from (19) as $\alpha = \log(a)/\log(d)$ etc.:

$$x = k_1 t^{\log(a)/\log(d)},\; y = k_2 t^{\log(b)/\log(d)},\; z = k_3 t^{\log(c)/\log(d)} \tag{24}$$

It should be noted that we cannot remove the common exponent $1/\log(d)$.

When the characteristic roots are all real and distinct this is a twisted curve.

For the case when two characteristic roots are equal, say a and b, we have

$$x = k_1 t^{\log(a)/\log(d)},\; y = k_2 t^{\log(a)/\log(d)},\; z = k_3 t^{\log(c)/\log(d)}$$

clearly showing that x and y are in a fixed ratio so that the path curve lies in a plane. An example is shown below where the continuous line is generated by the parametric equations and the crosses are produced by recursive application of the transformation matrix:

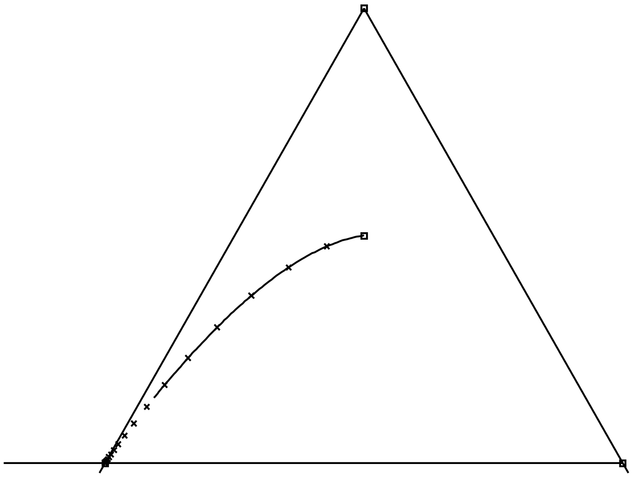

Figure 3

For a pair of complex conjugate roots $re^{\pm i\theta}$ we have

$$x = k_1 t^{(i\theta+\log(r))/\log(d)},\ \ y = k_2 t^{((\log(r)-i\theta)/\log(d)},\ \ z = k_3 t^{\log(c)/\log(d)} \qquad (25)$$

and as x and y are complex conjugate the real reference coordinates x_1 y_1 z_1 t_1 (say) are then

$x_1 = k\cos(\varepsilon)$, $y_1 = k\sin(\varepsilon)$ where $\varepsilon = \theta\log(t)/\log(d)$ and $k_1=k_2=kt^{-\log(r)/\log(d)}$

$z_1 = k_3 t_1^{\log(c)/\log(d)}$ as z and t are real

which gives a twisted spiral passing from the invariant points {1,±i,0,0} to {0,0,0,1} i.e. rotating about the real axis through the latter point. An example of a vortex is shown below, again with the parametric equations plotted as a continuous line and the recursive plot shown by crosses on the line. The right hand spiral shows the same plot viewed from the top:

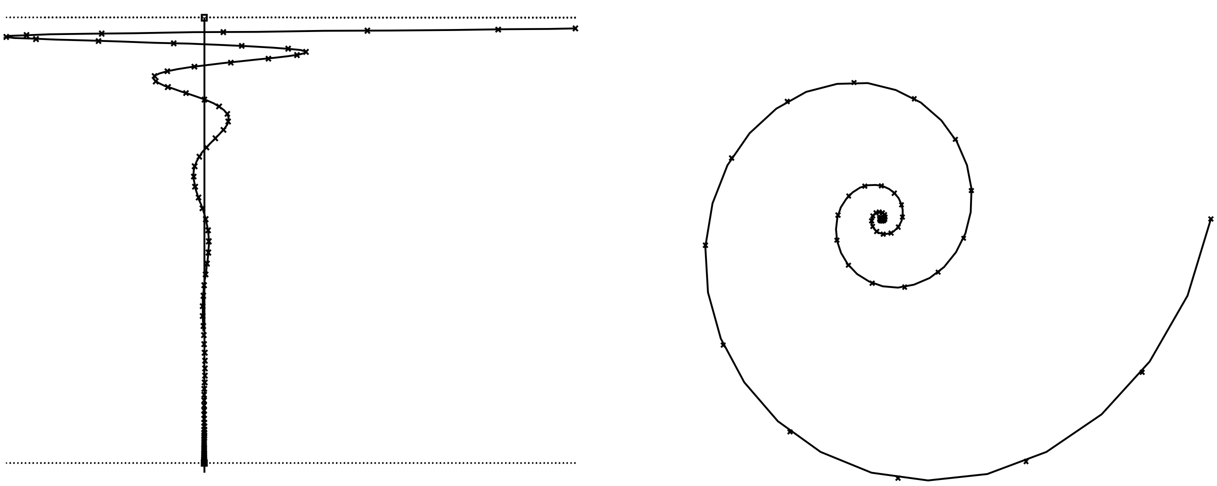

Figure 4

For two pairs of complex conjugate roots $ae^{\pm i\alpha}$ and $ce^{\pm i\beta}$ we have $x=re^{i\theta}$, $y=re^{-i\theta}$, $z=se^{i\phi}$, $t=se^{-i\phi}$ from which $d=t_2/t_1=e^{-i\delta}$ and hence $\log(d) = -i\delta$ where $\delta=\phi_2-\phi_1$ is a constant, so that from (24) we have

$$x = k_1t^{[\alpha+i \log(a)]/\delta},\ \ y = k_2t^{[-\alpha+i \log(a)]/\delta},\ \ z = k_3t^{[\beta+i \log(c)]/\delta}.$$

But as $t = se^{-i\phi}$, $x = k_1\, a^{\phi/\delta}s^{\alpha/\delta}\, e^{\,i[\alpha\phi + \log(a)\log(s)]/\delta}$, $y = k_2\, a^{\phi/\delta}s^{-\alpha/\delta}\, e^{\,i[-\alpha\phi + \log(a)\log(s)]/\delta}$

Changing to real coordinates x_1 y_1 z_1 t_1 and choosing k_1, k_2 such that $k=k_1s^{\,\alpha/\delta} =k_2s^{\,-\alpha/\delta}$ we get

$$x_1 = R\cos(\varepsilon),\ \ y_1 = R\sin(\varepsilon)\quad \text{where } R = ka^{\phi/\delta} \text{ and } \varepsilon= [\log(a)\log(s)+\alpha\phi]/\delta$$

and as $z_3 = k_3c^{\phi/\delta}s^{\beta/\delta}e^{i[\beta\phi+\log(c)\log(s)]/\delta} = se^{i\phi}$ for z_3 and t to be complex conjugates, we set $k_3=c=1$ and $\delta=\beta$ to give

$$z_1 = s\cos\phi,\ \ t_1 = s\sin\phi \text{ in real coordinates.}$$

In Cartesian coordinates we get the parametric equations in terms of ϕ:

$$x = (R/s)\cos(\varepsilon)/\sin\phi,\ \ y = (R/s)\sin(\varepsilon)/\sin\phi,\ \ z = \cot\phi$$

so that x and y vary sinusoidally while z repeatedly passes from 0 to ∞ as ϕ increases, as there are no real invariant points, the spiral repeatedly winding around and along the z-axis, varying in radius during each pass. An example is shown below, again from the side and the top:

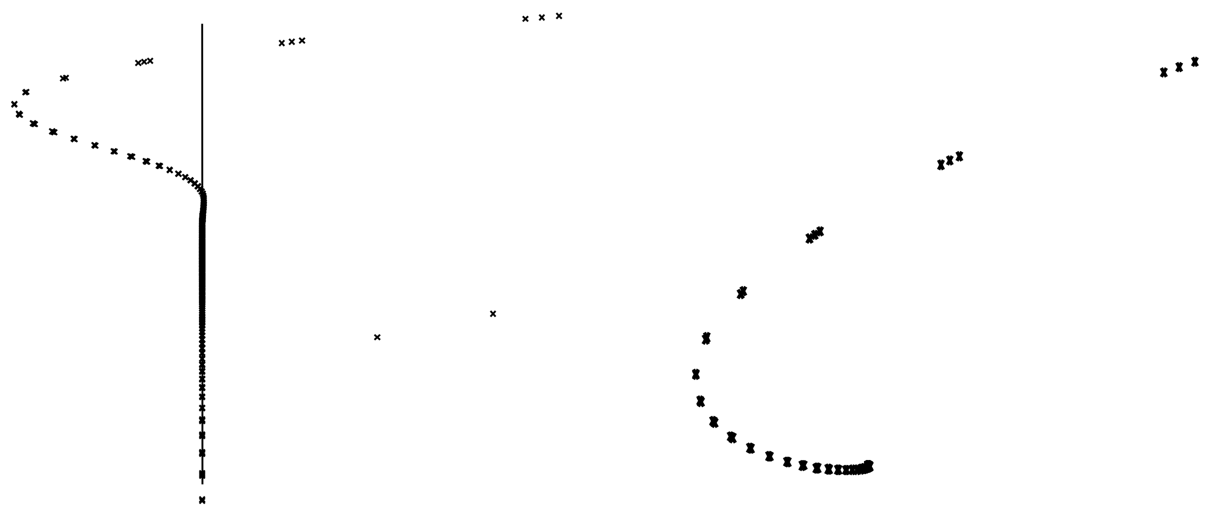

Figure 5

A comparison with the recursive case proved more difficult, but is shown below for the same plot for a small region, the parametric plot being shown by points:

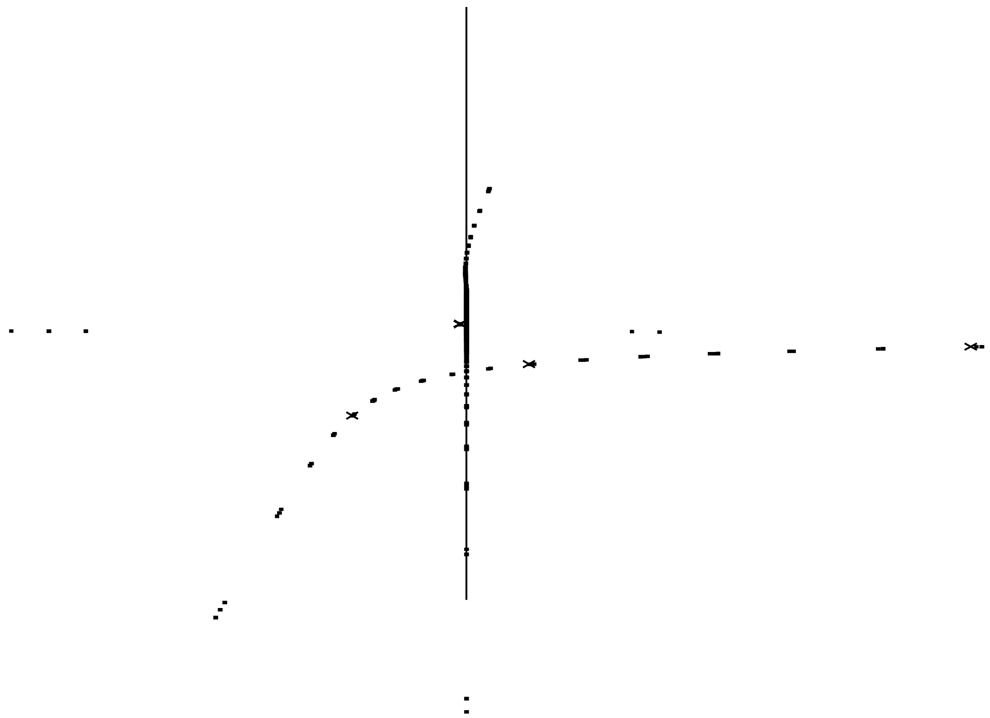

Figure 6

<u>Summary</u>

A two dimensional path curve has the equation (c.f. also Edwards *Projective Geometry* Rudolf Steiner Institute 1985)

$$x^u y^v z^w = k$$

A path-surface in three dimensions has the equation

$$x^u y^v z^w t^s = k$$

A path curve in three dimensions has the parametric equations

$$x = k_1 t^\alpha, \;\; y = k_2 t^\beta, \;\; z = k_3 t^\gamma$$

The path curve associated with a collineation with characteristic roots a b c d is

$$x = k_1 t^{\log(a)/\log(d)}, \;\; y = k_2 t^{\log(b)/\log(d)}, \;\; z = k_3 t^{\log(c)/\log(d)}$$