Path Curves Referred to Two-Line Tetrahedron

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Lawrence Edwards explains path curves in projective geometry (Edwards 1993) to which the reader is referred. Familiarity with homogeneous coordinates is assumed in what follows. Such coordinates are referred to an invariant tetrahedron which may be fully real, or may have two of its lines, points and planes imaginary, or have only two real lines and no real vertices or planes: a "two-line tetrahedron" (TLT). This article explores path curves referred to a TLT. A canonical linear transform referred to a TLT is of the form

$$T=\begin{bmatrix} r e^{i\mu} & & & \\ & s e^{i\nu} & & \\ & & r e^{-i\mu} & \\ & & & s e^{-i\nu} \end{bmatrix} \tag{1}$$

where the vertical z-axis is taken as one of the real lines of the reference tetrahedron, and the terms $r e^{\pm i\mu}$ transform wrt that axis, while the terms $s e^{\pm i\nu}$ relate to the other real line of the tetrahedron which will be taken as the line at infinity where planes orthogonal to the z-axis meet. The matrix for that tetrahedron is

$$M=\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ i & 0 & -i & 0 \\ 0 & i & 0 & -i \end{bmatrix} \tag{2}$$

so that the transform (1) is referred to it in canonical form. Multiplying a point in this coordinate system by (2) transforms it into a real Cartesian point. Conversely a Cartesian point is transformed into these coordinates by multiplying it by the inverse of (2), which is

$$M^{-1}=\frac{1}{2}\begin{bmatrix} 1 & 0 & -i & 0 \\ 0 & 1 & 0 & -i \\ 1 & 0 & i & 0 \\ 0 & 1 & 0 & i \end{bmatrix} \tag{3}$$

The purely Cartesian transformation of a point P={x_0 x_1 x_2 x_3}, which obviates the need to use complex coordinates in practical cases, is thus $MTM^{-1}P$:

$$\frac{1}{2}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ i & 0 & -i & 0 \\ 0 & i & 0 & -i \end{bmatrix}\begin{bmatrix} r e^{i\mu} & & & \\ & s e^{i\nu} & & \\ & & r e^{-i\mu} & \\ & & & s e^{-i\nu} \end{bmatrix}\begin{bmatrix} 1 & 0 & -i & 0 \\ 0 & 1 & 0 & -i \\ 1 & 0 & i & 0 \\ 0 & 1 & 0 & i \end{bmatrix}\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} x_0 r\cos\mu + x_2 r\sin\mu \\ x_1 s\cos\nu + x_3 s\sin\nu \\ -x_0 r\sin\mu + x_2 r\cos\mu \\ -x_1 s\sin\nu + x_3 s\cos\nu \end{bmatrix}$$

i.e. {x_0 x_1 x_2 x_3} is transformed into {y_0 y_1 y_2 y_3} by

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_0 r \cos\mu + x_2 r \sin\mu \\ x_1 s \cos\nu + x_3 s \sin\nu \\ -x_0 r \sin\mu + x_2 r \cos\mu \\ -x_1 s \sin\nu + x_3 s \cos\nu \end{bmatrix} \tag{4}$$

This enables the path curve to be plotted using real coordinates only, where y_j is the transform of x_j. If the transform is applied repeatedly to generate the curve recursively then (1) is raised to the power n for n iterations, so that r may be replaced by r^n, s by s^n, μ by $n\mu$ and ν by $n\nu$ in (4).

The following picture shows such a path curve with r=1.0001, s=1, $\mu=10^O$ and $\nu=0.2^O$. The centre is shown magnified on the right, where the start and end points of the portion plotted are seen. Note that such a path curve never approaches any fixed point so in principle infinitely may iterations are required to describe it fully. In practice a finite number must be taken.

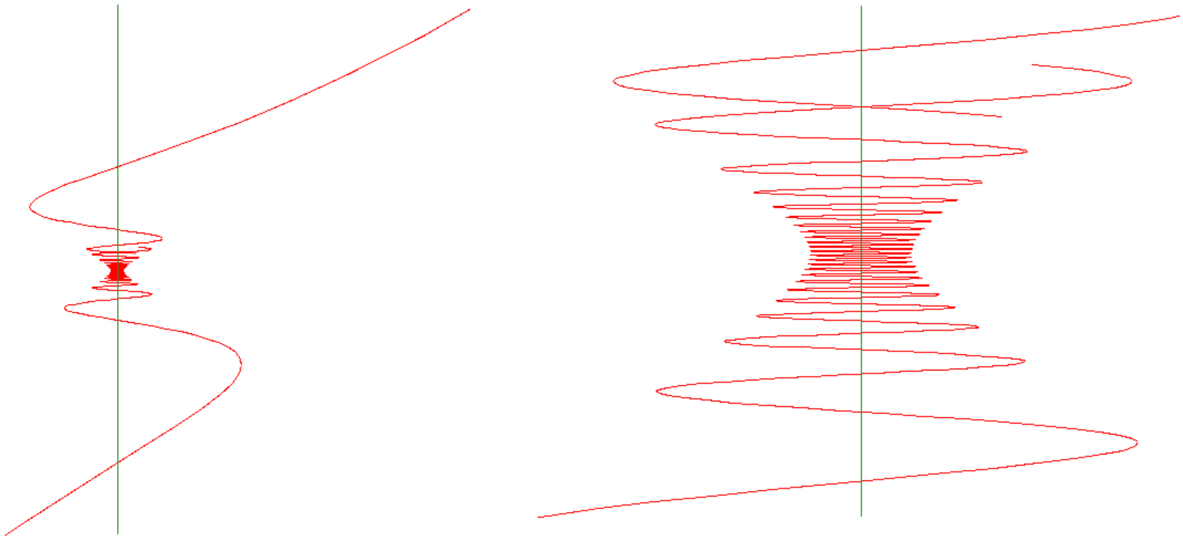

Figure 1

The picture below shows how one curve may envelope a surface about the z-axis. The parameters are r=0.9999, s=1, $\mu=44.5^o$ and $\nu=45.5^o$:

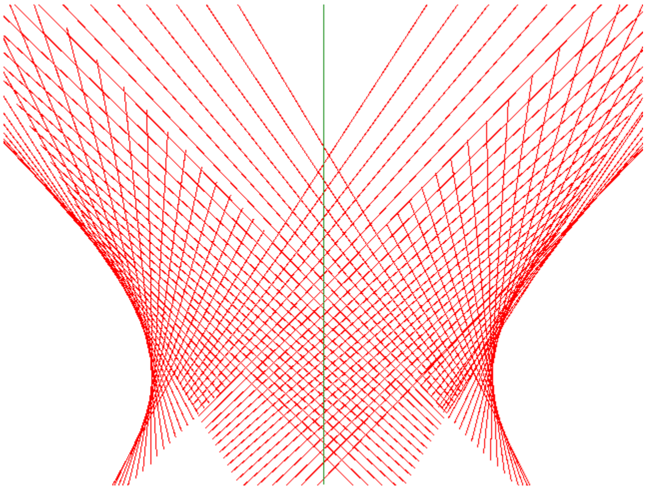

Figure 2

Decreasing r to 0.999 and increasing the number of points plotted shows the spiral nature of such a surface:

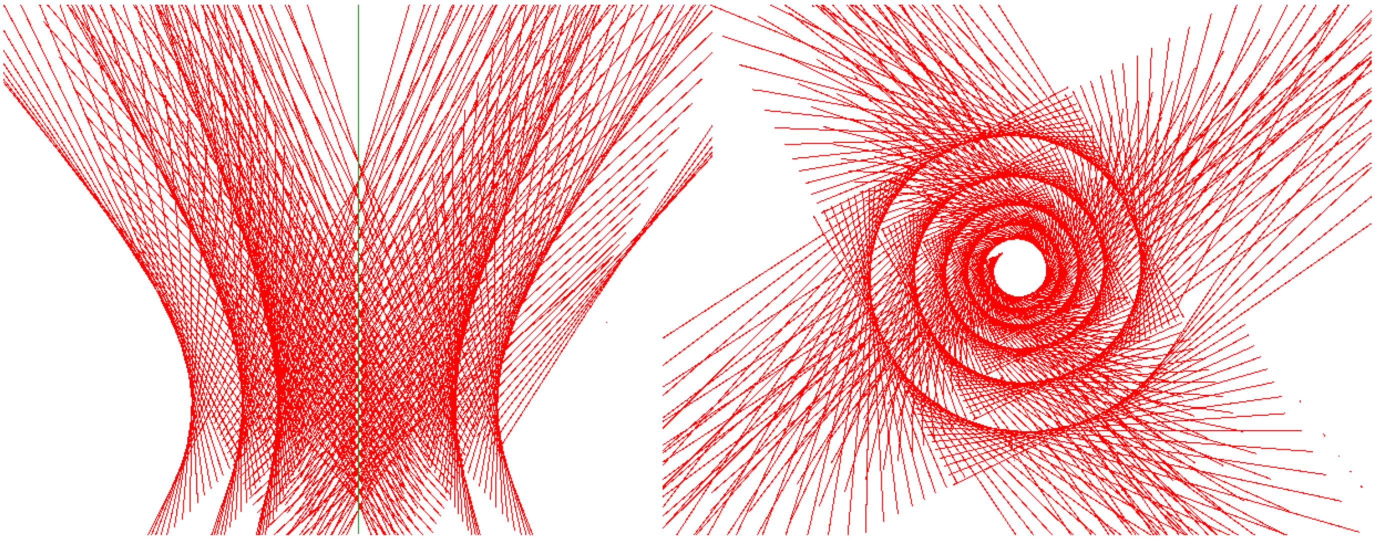

Figure 3

The left plot shows the view with the z-axis vertical while the right shows a view from the top, exhibiting the spiral nature of the surface. The picture shows just one path curve.

Rescaling using s=1, and taking $x_2=0$ to start in the x-z plane, and also setting $x_3=1$ for Cartesian coordinates, (4) becomes:

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_0 r^n \cos(n\mu) \\ x_1 \cos(n\nu) + \sin(n\nu) \\ -x_0 r^n \sin(n\mu) \\ \cos(n\nu) - x_1 \sin(n\nu) \end{bmatrix} \tag{5}$$

If further $n\nu=2\pi$ then

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_0 r^n \cos(n\mu) \\ x_1 \\ -x_0 r^n \sin(n\mu) \\ 1 \end{bmatrix} \tag{6}$$

so that the point returns to the same horizontal level after n iterations since $y_1/y_3=x_1$. The radius of the point - i.e. its horizontal distance from the vertical z-axis - is $r^n x_0$ at an angle $n\mu$ to the initial horizontal radial vector. Thus repeating for another n iterations gives $r^{2n}x_0$ at an angle $2n\mu$ and so on, so these points n iterations apart lie on a horizontal logarithmic spiral, as illustrated in Figure 3. Since x_1 may be chosen arbitrarily to give the height of the horizontal plane, the surface meets all such planes in identical logarithmic spirals, but differently orientated about the z-axis.

If instead $x_1=0$ to start in the x-y plane, together with $s=x_3=1$, then for n transforms (4) becomes

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_0 r^n \cos(n\mu) + x_2 r^n \sin(n\mu) \\ \sin(n\nu) \\ -x_0 r^n \sin(n\mu) + x_2 r^n \cos(n\mu) \\ \cos(n\nu) \end{bmatrix}$$

If further $n_0\mu=2\pi$ then for n a multiple of n_0

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_0 r^n \\ \sin(n\nu) \\ x_2 r^n \\ \cos(n\nu) \end{bmatrix} \tag{7}$$

so that the points for n a multiple of n_0 all lie in a fixed vertical plane between the x- and y-axes, with $R=r^nR_0/\cos(n\nu)$ where $R_0=\sqrt{x_0^2+x_2^2}$, and $Z=\tan(n\nu)$ in that plane. Assuming $\cos(n\nu)\neq 0$ and eliminating ν using $t=\tan(n\nu)=Z$ gives

$$R=\pm r^n R_0\sqrt{1+t^2} \quad \text{i.e.} \quad \frac{R^2}{r^{2n}R_0^2}-Z^2=1 \tag{8}$$

Without r^{2n} this would represent a hyperbola. If r is close to 1 (with s=1) then the radius changes slowly so that $\tan(n\nu)$ traverses infinity many times as r^{2n} increases if ν is relatively large, giving many vertical passes, the profile being close to a hyperbolic curve. That yields surfaces such as in Figures 2 and 3 where $\nu=45.5^o$ and r=0.999. Such surfaces also arise if μ and ν are nearly but not quite equal, even if ν is small, but r must be carefully chosen close to 1 to obtain a full spiral. Note that the assumption that 2π is an exact multiple of μ does not negate the generality of this result as raising (1) to a suitable power can change μ to an integral fraction of 2π without changing the overall path curve.

If r differs appreciably from 1 then the curve may make only one vertical traverse before r^{2n} brings it close to the vertical axis (r<1) or out to the axis at infinity (r>1). If r<1 then R increases more slowly than for the hyperbola in (8), giving a vortex-like profile. An example with r=0.95, s=1, $\mu=10^o$ and $\nu=-0.2^o$ is shown below:

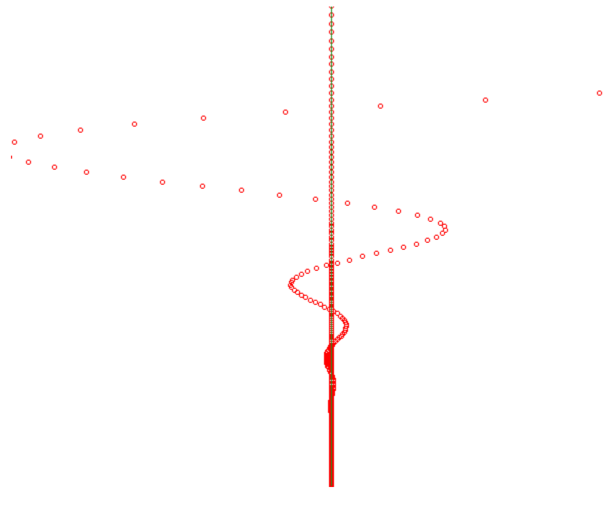

Figure 4

The vertical displacement is small because ν is small and the rotary displacement is larger because μ is relatively large. The second vertical pass is virtually on the axis. There are n_0=360/10=36 points between maxima on the left side (and on the right).

The above observations suggest that a parameter λ be defined similar to that for path curves related to a semi-imaginary tetrahedron used by Edwards i.e.

$$\lambda=\frac{\log(r/s)}{\nu} \tag{9}$$

which controls the profile such that as λ increases the profile moves away from a hyberbolic towards a "vortical" form. The steepness of the path curve spiral is controlled by the ratio of ν to μ so the parameter ϵ is defined as

$$\epsilon=\frac{\nu}{\mu} \tag{10}$$

In Figure 1: ν=0.2°, λ=0.0286 and ϵ=0.02; in Figure 3: ν=45.5°, λ=-0.00126 and ϵ=1.0225 and in Figure 4 ν=-0.2°, λ=14.69 and ϵ=-0.02 (noting that ν is in radians in (9)). ν is taken as the third parameter rather than μ so that calculating μ from ϵ and r/s from ν and λ ensures that the sign of epsilon determines the sense of the spiral. The horizontal cross-sections are logarithmic spirals with equation $R=Ae^{\theta\cot\alpha}$. Referring to (6) two instances of this equation are

$$r^n R_0=A\,e^{n\,\nu\cot\alpha}$$
$$r^{2n} R_0=A\,e^{2n\,\nu\cot\alpha}$$

and dividing the second by the first gives

$$\cot\alpha=\frac{\log(r/s)}{\nu}=\lambda \tag{11}$$

allowing for $s\neq1$. Thus λ determines the constant angle α of the logarithmic spirals and their sense. If λ is very small a surface is obtained, whereas if it is large then α is small and the spiral winds rapidly into the axis.

If r/s differs appreciably from 1 then the curve becomes less well defined visually, but generally this does not arise if it is defined by the above parameters.

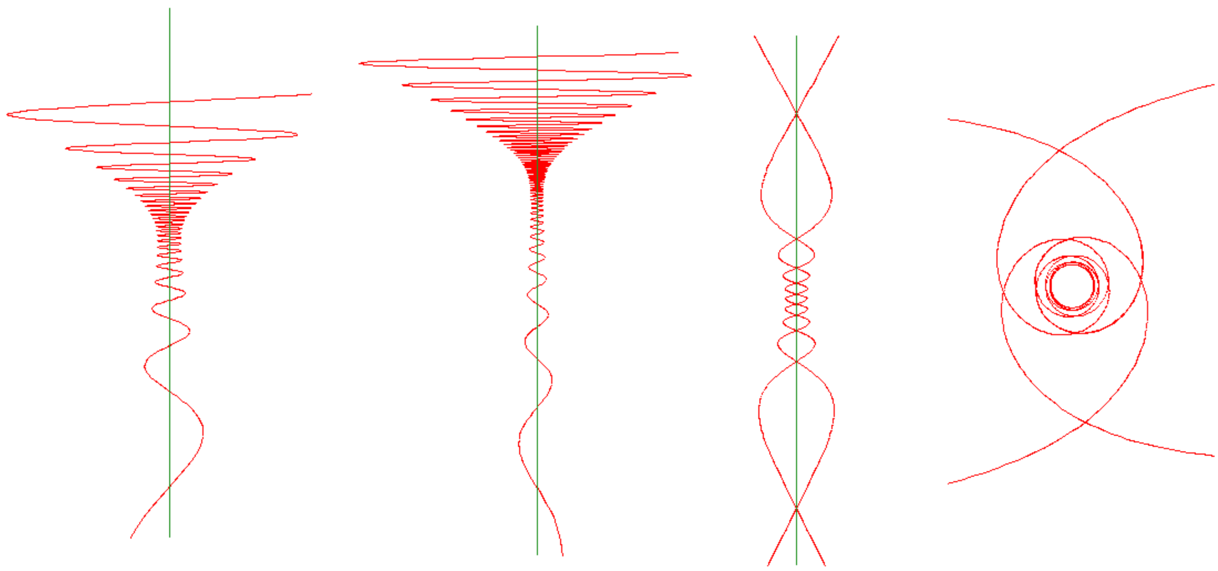

Figure 5

The figure on the left is for λ=0.832, ϵ=-0.02 and ν=-0.2^{o} which is like Lawrence Edward's "chalice" form (Edwards 1993). The next similar figure with λ=1.147, ϵ=-0.01 and ν=-0.2^{o} shows a relation between the earthly vertical and the cosmic "circling round". The figure on the right, with λ=0, ϵ=0.1 and ν=-0.25^{o}, is shown in elevation and on the far right from the top. It has a hyperbolic profile as λ=0 implies r/s=1 and hence r^n=1 in (8).

Conclusion

Path curves referred to a two-line tetrahedron may readily be studied in the real domain using (4). The definition of λ and ϵ in (9) and (10) enables a straight forward control of these curves. An interesting result is that a single path curve may describe a spiral surface.

Bibliography

Edwards, Lawrence, *The Vortex of Life* Floris Press, Edinburgh 1993.