Using Plücker Line Coordinates

The practical difficulty encountered with line coordinates in space is that given such a set it is not intuitively easy to see whether a point lies on the line, or a plane contains it, or where it intersects a given plane. How does one define a variable point or plane on the line? Solutions for these problems will be given here. Even plane coordinates are less than obvious, so we start with them.

We use Γ to denote the plane at infinity, and $\Psi \equiv x_0^2+x_1^2+x_2^2=0=x_3$ to denote the absolute conic in Γ.

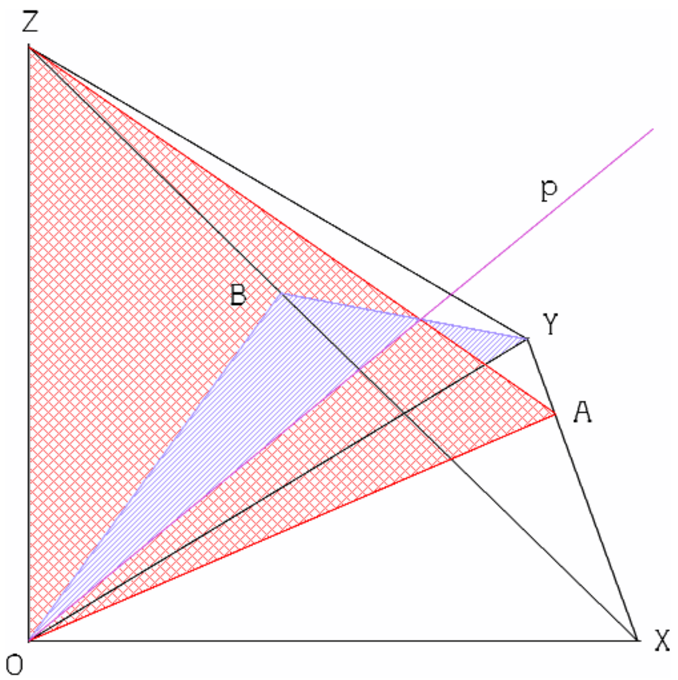
Plane Coordinates

Consider a plane {u v w k}: it intersects the x-axis in a point {x 0 0 1} so we have ux+k=0 i.e. the plane intersects the x-axis at x=-k/u. Similarly for the other axes:

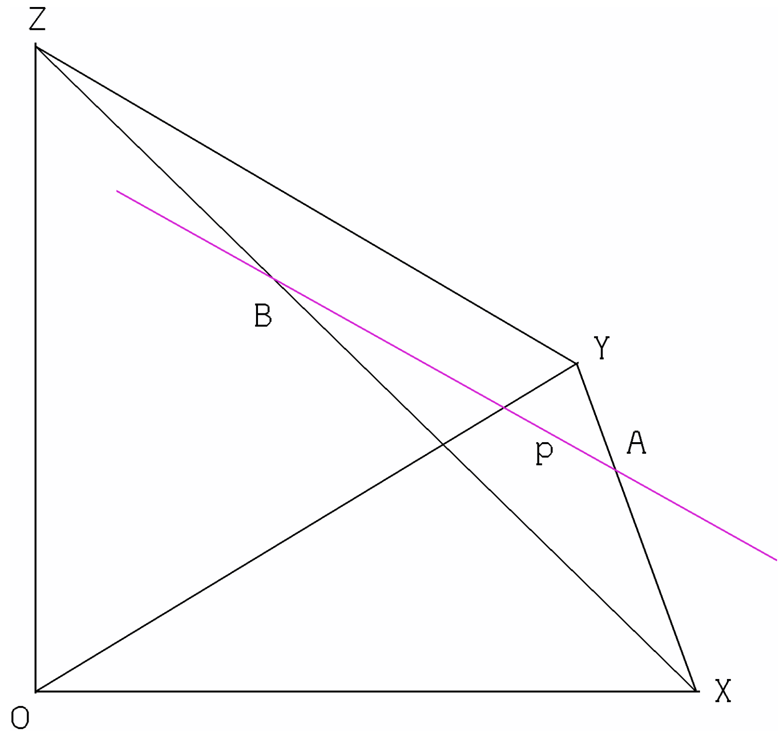
$$\begin{aligned} x &= -\frac{k}{u} \\ y &= -\frac{k}{v} \\ z &= -\frac{k}{w} \end{aligned} \qquad (1)$$

which enables the plane to be visualised from its coordinates.

Note that the plane (u v w 0) lies in the origin.



If three coordinates of a point are known, say x_0 x_1 and x_2, then the point lies on a line through a vertex of the invariant tetrahedron. In this example the ratio between x_0 and x_1 determines a plane such as the red plane OZA, so that varying the other two coordinates gives points in that plane. The ratio between x_0 and x_2 determines a plane such as the blue one, OYB. Thus the point lies on the line p where those planes intersect.



Dually if three coordinates of a plane are known, say u_0 u_1 and u_2, then the ratio of u_0 to u_1 determines a point such as A on the line XY, and that of u_0 to u_2 a point B on XZ. Varying u_3 gives an axial pencil in the line p=AB.

Plücker coordinates

These are defined initially in terms of the coordinates of two points $X=\{x_0\ x_1\ x_2\ x_3\}$ and $Y=\{y_0\ y_1\ y_2\ y_3\}$ by

$$P=\{\ x_0 y_1 - x_1 y_0\ ,\ x_0 y_2 - x_2 y_0\ ,\ x_0 y_3 - x_3 y_0\ ,\ x_1 y_2 - x_2 y_1\ ,\ x_3 y_1 - x_1 y_3\ ,\ x_2 y_3 - x_3 y_2\ \} \qquad \text{(A)}$$

which are expressed as $p_{jk} = \{\ p_{01}\ p_{02}\ p_{03}\ p_{12}\ p_{31}\ p_{23}\ \}$. Any other two points on the line XY are

$$\{\ x_0+\lambda y_0\ x_1+\lambda y_1\ x_2+\lambda y_2\ x_3+\lambda y_3\ \} \text{ and } \{\ x_0+\mu y_0\ x_1+\mu y_1\ x_2+\mu y_2\ x_3+\mu y_3\ \}$$

and for these points we have

$$p_{01} = (x_0+\lambda y_0)(x_1+\mu y_1) - (x_1+\lambda y_1)(x_0+\mu y_0) = (x_0 y_1 - x_1 y_0)(\lambda+\mu)$$

and similarly for the other p_{jk} so that the common factor $(\lambda+\mu)$ may be cancelled as we are using homogeneous coordinates, leaving us with (A) i.e. the ratios of the p_{jk} are the same no matter which two points X Y are chosen on the line. Hence the numbers p_{jk} may be thought of as coordinates of the line independently of any particular points. We cannot choose any six numbers for the p_{jk}, and the Plücker condition for them to represent a line is

$$p_{01}p_{23} + p_{02}p_{31} + p_{03}p_{12} = 0 \qquad \text{(B)}$$

which is readily verified to be satisfied by (A).

Suppose two lines have the point X in common, and let one join X and Y and the other X and Z. Then the lines are

$$p_{jk} = \{ x_0 y_1 - x_1 y_0 , x_0 y_2 - x_2 y_0 , x_0 y_3 - x_3 y_0 , x_1 y_2 - x_2 y_1 , x_3 y_1 - x_1 y_3 , x_2 y_3 - x_3 y_2 \}$$

and

$$q_{jk} = \{ x_0 z_1 - x_1 z_0 , x_0 z_2 - x_2 z_0 , x_0 z_3 - x_3 z_0 , x_1 z_2 - x_2 z_1 , x_3 z_1 - x_1 z_3 , x_2 z_3 - x_3 z_2 \}$$

and we find that

$$p_{01}q_{23}+p_{02}q_{31}+p_{03}q_{12}+p_{12}q_{03}+p_{31}q_{02}+p_{23}q_{01} = 0 \qquad \text{(C)}$$

which is often denoted by

$$\Omega_{pq}=0 \qquad \text{(C1)}$$

which is the Plücker condition for two lines to intersect.

<u>Special Lines</u>

The coordinate lines joining the points X={1 0 0 0}, Y={0 0 1 0}, Z={0 1 0 0} and O={0 0 0 1} are

XO = {0 0 1 0 0 0} $\quad p_{03}$ because the only non-zero coords are x_0 and o_3
YO = {0 0 0 0 0 1}
ZO = {0 0 0 0 1 0}
XY = {0 1 0 0 0 0}
XZ = {1 0 0 0 0 0}
YZ = {0 0 0 1 0 0}

If p_{jk} intersects such an edge q_{jk} of the tetrahedron then only one of the q_{jk} is non-zero, so one of the p_{jk} is zero to satisfy (C). It will be the "polar" coordinate e.g. for the x-axis q_{03}=1 so p_{12}=0.
Thus $\{p_{01}\ p_{02}\ 0\ p_{12}\ p_{31}\ p_{23}\}$ meets XO. Similarly for the other sides.

If p_{jk} lies in a plane of the tetrahedron then it possesses three zeros:

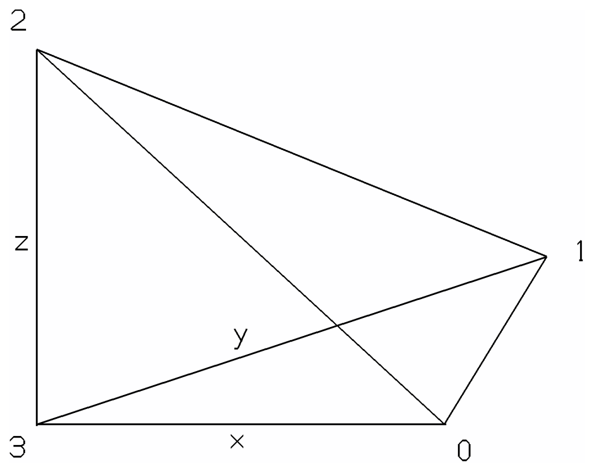
in XYZ: $\{p_{01}\ p_{02}\ 0\ p_{12}\ 0\ 0\}$ satisfies (C) for XY, XZ and YZ
in XZO: $\{p_{01}\ 0\ p_{03}\ 0\ p_{31}\ 0\}$ satisfies (C) for XZ, XO and ZO
in XYO: $\{0\ p_{02}\ p_{03}\ 0\ 0\ p_{23}\}$ satisfies (C) for XY, XO and YO
in YZO: $\{0\ 0\ 0\ p_{12}\ p_{31}\ p_{23}\}$ satisfies (C) for ZO, YO and YZ

If the line meets a vertex then it also possesses three zeros:

in O: $\{0\ 0\ p_{03}\ 0\ p_{31}\ p_{23}\}$ satisfies (C) for XO, YO and ZO
in X: $\{p_{01}\ p_{02}\ p_{03}\ 0\ 0\ 0\}$ satisfies (C) for XZ, XY and XO
in Y: $\{0\ p_{02}\ 0\ p_{12}\ 0\ p_{23}\}$ satisfies (C) for YO, XY and YZ
in Z: $\{p_{01}\ 0\ 0\ p_{12}\ p_{31}\ 0\}$ satisfies (C) for XZ, ZO and YZ

With four zeros it meets a vertex and lies in one of the adjacent sides. Note that of the 15 combinations of two ones amongst six coordinates, three do not satisfy (B). Similarly in the above cases.

This is summed up in the following diagram:



The x-axis OX is the line joining vertices 3 and 0. This line has all coordinates zero except p_{03}. Similarly the y-axis OY has all coordinates zero except p_{03}, etc..
The line in a plane has those coordinates zero which contain the number of the opposite vertex.
The line meeting a vertex has those coordinates non-zero which contain the number for that vertex.
The line meeting an edge has one coordinate zero given by the numbers of the opposite (skew) edge.

Intersection of a Line with the Plane at Infinity

The line joining the points A={a b c 0} in the plane at infinity and E={e f g h} is by (A)

$$\{ be-af, ce-ag, -ah, cf-bg, bh, -ch \}$$

so since $\{a\ b\ c\ 0\}\equiv\{ah\ bh\ ch\ 0\}$ for homogeneous coordinates, we have

$$A = \{-p_{03}\ p_{31}\ -p_{23}\ 0\} \tag{D}$$

Thus varying p_{01}, p_{02}, p_{12} gives lines in A i.e. parallel lines. Note that only two of p_{01}, p_{02}, p_{12} may be chosen freely as the third is then determined by (B), so that there are ∞^2 such parallel lines.

Intersection of a Line with a general Plane

Suppose the plane is $U=u_j$ and a variable point $X=x_j$ is on the line p_{jk}, joined to the point at infinity $Y=y_j$ on p_{jk}, then we seek X in U. Using (A), $p_{jk}=\{x_0y_1-x_1y_0, x_0y_2-x_2y_0, x_0y_3-x_3y_0, ...\}=\{p_{01}, p_{02}, p_{03} ...\}$ and noting that $Y= \{-p_{03}\ p_{31}\ -p_{23}\ 0\}$ from (D) we have

$$x_1=\frac{x_0y_1-p_{01}}{y_0}=\frac{p_{01}-x_0p_{31}}{p_{03}} \tag{E}$$

$$x_2=\frac{x_0y_2-p_{02}}{y_0}=\frac{x_0p_{23}+p_{02}}{p_{03}} \tag{F}$$

$$x_3=\frac{x_0y_3-p_{03}}{y_0}=1 \tag{G}$$

Now for X to lie in U we have $\Sigma x_ju_j=0$ i.e.

$$x_0 u_0 + \frac{p_{01} - x_0 p_{31}}{p_{03}} u_1 + \frac{x_0 p_{23} + p_{02}}{p_{03}} u_2 + u_3 = 0$$

giving

$$x_0 = \frac{p_{01} u_1 + p_{02} u_2 + p_{03} u_3}{-p_{03} u_0 + p_{31} u_1 - p_{23} u_2} = \frac{N}{D} \text{ , say} \tag{H}$$

Substituting in (E) (F) and (G) we then get

$$\begin{aligned} X &= \left\{ \frac{N}{D} \, , \, \frac{p_{01}}{p_{03}} - \frac{N}{D} \frac{p_{31}}{p_{03}} \, , \, \frac{N}{D} \frac{p_{23}}{p_{03}} + \frac{p_{02}}{p_{03}} \, , \, 1 \right\} \\ &= \{ N p_{03} \, , \, D p_{01} - N p_{31} \, , \, N p_{23} + D p_{02} \, , \, D p_{03} \} \\ &= \{ N \, , \, p_{12} u_2 - p_{01} u_0 - p_{31} u_3 \, , \, p_{23} u_3 - p_{02} u_0 - p_{12} u_1 \, , \, D \} \end{aligned}$$

using (B) in the simplification. Thus expanding N and D we have

$$\begin{aligned} X = \{ \quad & \qquad\qquad\quad p_{01} u_1 + p_{02} u_2 + p_{03} u_3 \, , \\ & -p_{01} u_0 \qquad\quad + p_{12} u_2 - p_{31} u_3 \, , \\ & -p_{02} u_0 - p_{12} u_1 \qquad\quad + p_{23} u_3 \, , \\ & -p_{03} u_0 + p_{31} u_1 - p_{23} u_2 \qquad\qquad \} \end{aligned}$$

or in matrix form

$$X = \begin{bmatrix} 0 & p_{01} & p_{02} & p_{03} \\ -p_{01} & 0 & p_{12} & -p_{31} \\ -p_{02} & -p_{12} & 0 & p_{23} \\ -p_{03} & p_{31} & -p_{23} & 0 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{I}$$

For the plane at infinity {0 0 0 1} we recover (D), albeit with the signs reversed. We deduce the intercepts with the coordinate planes:

XYO plane	$\{p_{01}\ 0\ -p_{12}\ p_{31}\}$	(J)
XZO plane	$\{p_{02}\ p_{12}\ 0\ -p_{23}\}$	
YZO plane	$\{0\ \ -p_{01}\ -p_{02}\ -p_{03}\}$	
infinite plane	$\{p_{03}\ -p_{31}\ p_{23}\ 0\}$	

(I) reveals an interesting relationship between a line and a null polarity, for X is the pole of U wrt the null-polarity matrix defined by p_{jk}. The line is the axis of the associated special linear complex. For every plane it yields the polar point, identical of course with the point of intersection of the axis and the plane.

Another way of expressing this beautiful relationship is that a line meets a plane in the point polar to that plane wrt the null polarity with the line as axis.

Skew-Orthogonal Lines

If p_{jk} and q_{jk} are skew-orthogonal then they meet the plane at infinity Γ in conjugate points, i.e. $\{p_{03}\ -p_{31}\ p_{23}\}$ and $\{q_{03}\ -q_{31}\ q_{23}\}$ in Γ are conjugate wrt the absolute conic $\Psi \equiv x_0^2+x_1^2+x_2^2=0=x_3$. Now the point y_j in Γ is conjugate to x_j wrt Ψ if $x_0y_0+x_1y_1+x_2y_2=0$ from the equation for Ψ, so $\{p_{03}\ -p_{31}\ p_{23}\}$ and $\{q_{03}\ -q_{31}\ q_{23}\}$ are conjugate if

$$p_{03}q_{03}+p_{31}q_{31}+p_{23}q_{23} = 0 \qquad \text{(K)}$$

which is the condition for p_{jk} and q_{jk} to be skew-orthogonal.

A Variable Point on a Line

If x_j is a variable point on the line p_{jk} then as the line joins the points $\{p_{03}\ -p_{31}\ p_{23}\ 0\}$ and $\{0\ -p_{01}\ -p_{02}\ -p_{03}\}$ we have

$$x_j = \{\lambda p_{03}\ ,\ -\lambda p_{31}-\mu p_{01}\ ,\ \lambda p_{23}-\mu p_{02}\ ,\ -\mu p_{03}\} \qquad \text{(M)}$$

The problem with this is that it does not include p_{12}.
This fails if all but p_{12} are zero, in which case the line is YZ at infinity and $x_j = \{0\ \lambda\ \mu\ 0\}$.
An alternative form if p_{03} is not zero is obtained by joining $\{p_{01}\ 0\ -p_{12}\ p_{31}\}$ and $\{p_{02}\ p_{12}\ 0\ -p_{23}\}$:

$$x_j = \{\lambda p_{01}+\mu p_{02}\ ,\ \mu p_{12}\ ,\ -\lambda p_{12}\ ,\ \lambda p_{31}-\mu p_{23}\}$$

Orthogonal Plane and Line

A plane u_j meets Γ in the line $\{u_0\ u_1\ u_2\}$ while a line p_{jk} meets Γ in the point $\{-p_{03}\ p_{31}\ -p_{23}\}$ by (D). Thus the two are orthogonal if

$$p_{03}u_0-p_{31}u_1+p_{23}u_3 = 0 \qquad \text{(MM)}$$

Dual View

Two planes u_j v_j intersect in a line, and we may form the similar quantities as (A), say π_{jk} in plane coordinates, and the line condition then follows as before. If we take the plane u_j and Γ ={0 0 0 1} then

$$\pi_{jk}=\{0\ 0\ u_0\ 0\ -u_1\ u_2\}$$

whereas if we take two points $\{x_0\ x_1\ x_2\ 0\}$ and $\{y_0\ y_1\ y_2\ 0\}$ in Γ and on this line, the p_{jk} are

$$p_{jk}=\{x_0y_1-x_1y_0\quad x_0y_2-x_2y_0\quad 0\quad x_1y_2-x_2y_1\quad 0\quad 0\}$$

so the π_{jk} are not the same as the p_{jk}. Now take the planes {1 0 0 0} and {0 1 0 0}: the only surviving term is π_{01}. If x_j lies in both these planes then $x_j=\{0\ 0\ x_2\ x_3\}$, and if v_j also lies in them then $v_j=\{0\ 0\ v_2\ v_3\}$. The line joining these points is $p_{jk}=\{0\ 0\ 0\ 0\ 0\ p_{23}\}$. Repeating for the other five special cases we find that q_{jk} is "polar" to p_{jk} giving us

$$\pi_{01} : \pi_{02} : \pi_{03} : \pi_{12} : \pi_{31} : \pi_{23} = p_{23} : p_{31} : p_{12} : p_{03} : p_{02} : p_{01} \tag{O}$$

In order to use all our previous results, we must make this reversal. However the dual results also now follow.

By symmetry (B) and (C) apply to the plane based coordinates without the need for (O).

We now state the significant duals where the π_{jk} have been reversed into p_{jk} according to (O)

Dual to (I) we have the plane determined by a line and a point as

$$U=\begin{bmatrix} 0 & p_{23} & p_{31} & p_{12} \\ -p_{23} & 0 & p_{03} & -p_{02} \\ -p_{31} & -p_{03} & 0 & p_{01} \\ -p_{12} & p_{02} & -p_{01} & 0 \end{bmatrix}\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{P}$$

Thus the plane in the origin and the line is $\{p_{12}\ -p_{02}\ p_{01}\ 0\}$. (P1)

Dual to (M) we have a variable plane in a line:

$$u_j = \{\lambda p_{12} ,\ -\lambda p_{02}-\mu p_{23} ,\ \lambda p_{01}-\mu p_{31} ,\ -\mu p_{12}\} \tag{Q}$$

If all the p_{jk} but p_{03} are zero then the plane is $\{\ \lambda\ 0\ 0\ \mu\}$.
The inner product of (M) and (Q) is zero, verifying the point and plane are incident.

Projection of a Line into the Plane at Infinity

If a line p_{jk} is projected from the origin into $\Gamma=\{0\ 0\ 0\ 1\}$ by means of the plane (P1) then that plane $\{p_{12}\ {-p_{02}}\ p_{01}\ 0\}$ meets Γ in the line $\pi_{jk}=\{0\ 0\ p_{12}\ 0\ p_{02}\ p_{01}\}$. The polar to π_{jk} using (O) is $q_{jk}=\{p_{01}\ p_{02}\ 0\ p_{12}\ 0\ 0\}$. This is correctly the join of two points u_j and v_j in Γ with $u_3=v_3=0$. Now the two-dimensional coordinates of the line joining u_j and v_j are $\{u_1v_2-u_2v_1\ ,\ u_2v_0-u_0v_2\ ,\ u_0v_1-u_1v_0\} = \{p_{12}\ {-p_{02}}\ p_{01}\}$. Hence the line in the plane at infinity into which p_{jk} is projected from the origin is

$$\{p_{12}\ {-p_{02}}\ p_{01}\} \tag{S}$$

Conversely a line $\{u\ v\ w\}$ in Γ has the Plücker coordinates q_{jk} above:

$$\{w\ {-v}\ 0\ u\ 0\ 0\} \tag{T}$$

Projection of a Point into the Plane at Infinity

As a check, the line through the point x_j and the origin is $p_{jk}=\{0\ 0\ x_0\ 0\ {-x_1}\ x_2\}$.
Using (D) this meets Γ in $\{-p_{03}\ p_{31}\ {-p_{23}}\ 0\}=\{x_0\ x_1\ x_2\ 0\}$ as expected.

Point of Intersection of two Lines

To find the point of intersection of p_{jk} and q_{jk} we choose a point x_j not on either line nor in their common plane, so that the plane common to p_{jk} and x_j is

$$u_j=\begin{bmatrix} 0 & p_{23} & p_{31} & p_{12} \\ -p_{23} & 0 & p_{03} & -p_{02} \\ -p_{31} & -p_{03} & 0 & p_{01} \\ -p_{12} & p_{02} & -p_{01} & 0 \end{bmatrix}\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}=\boldsymbol{Px}\ \text{, say}$$

Then the required point of intersection y_j of the lines is that of q_{jk} and u_j which using (I) is

$$y_j=\begin{bmatrix} 0 & q_{01} & q_{02} & q_{03} \\ -q_{01} & 0 & q_{12} & -q_{31} \\ -q_{02} & -q_{12} & 0 & q_{23} \\ -q_{03} & q_{31} & -q_{23} & 0 \end{bmatrix}\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix}=\boldsymbol{Qu}$$

This assumes the lines do in fact intersect; if necessary use (C) to test that first.
Hence

$$y_j=\begin{bmatrix} 0 & q_{01} & q_{02} & q_{03} \\ -q_{01} & 0 & q_{12} & -q_{31} \\ -q_{02} & -q_{12} & 0 & q_{23} \\ -q_{03} & q_{31} & -q_{23} & 0 \end{bmatrix}\begin{bmatrix} 0 & p_{23} & p_{31} & p_{12} \\ -p_{23} & 0 & p_{03} & -p_{02} \\ -p_{31} & -p_{03} & 0 & p_{01} \\ -p_{12} & p_{02} & -p_{01} & 0 \end{bmatrix}\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{U}$$

Now y_j is independent of x_j provided x_j is not in the plane v_j determined by p_{jk} and q_{jk}, so we need to select x_j accordingly. As a trial we choose the unit point $\{1,1,1,1\}$ which ensures that all coordinates of p_{jk} and q_{jk} are included (which is important as a line could have only one non-zero coordinate) giving:

$$y_j = \begin{bmatrix} & q_{01} & q_{02} & q_{03} \\ -q_{01} & & q_{12} & -q_{31} \\ -q_{02} & -q_{12} & & q_{23} \\ -q_{03} & q_{31} & -q_{23} & \end{bmatrix} \begin{bmatrix} p_{23}+p_{31}+p_{12} \\ -p_{23}+p_{03}-p_{02} \\ -p_{31}-p_{03}+p_{01} \\ -p_{12}+p_{02}-p_{01} \end{bmatrix} \tag{V}$$

If {1,1,1,1} lies in v_j then some other point with four non-zero coordinates must be tried. The most elegant and safest approach is to find their common plane using (W) below, which cannot be {1,1,1,1}, and treat the result as a point which then cannot lie in that plane.

Athough this is not as simple as could be achieved using (M), it has the virtue that all possible lines are covered without exception, as all possible products $p_{jk}q_{lm}$ are present. Thus it is guaranteed to work e.g. in a computer program. For example, the special lines {1 0 0 0 0 0} and {0 1 0 0 0 0} meet in

$$\begin{bmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

which is correct.

Dually, the plane common to two lines is

$$\begin{aligned} u_j &= \begin{bmatrix} & \pi_{01} & \pi_{02} & \pi_{03} \\ -\pi_{01} & & \pi_{12} & -\pi_{31} \\ -\pi_{02} & -\pi_{12} & & \pi_{23} \\ -\pi_{03} & \pi_{31} & -\pi_{23} & \end{bmatrix} \begin{bmatrix} \mu_{23}+\mu_{31}+\mu_{12} \\ -\mu_{23}+\mu_{03}-\mu_{02} \\ -\mu_{31}-\mu_{03}+\mu_{01} \\ -\mu_{12}+\mu_{02}-\mu_{01} \end{bmatrix} \\ &= \begin{bmatrix} & q_{23} & q_{31} & q_{12} \\ -q_{23} & & q_{03} & -q_{02} \\ -q_{31} & -q_{03} & & q_{01} \\ -q_{12} & q_{02} & -q_{01} & \end{bmatrix} \begin{bmatrix} p_{01}+p_{02}+p_{03} \\ -p_{01}+p_{12}-p_{31} \\ -p_{02}-p_{12}+p_{23} \\ -p_{03}+p_{31}-p_{23} \end{bmatrix} \end{aligned} \tag{W}$$

If the plane {1,1,1,1} lies in their common point then some other plane with four non-zero coordinates must be tried. The most elegant and safest approach is to find their common point using (V), which cannot be {1,1,1,1}, and treat the result as a plane which then cannot contain that point.

Condition for a Point to lie on a Line

If x_j lies on p_{jk} then the plane U in (P) is singular i.e.

$$\begin{bmatrix} 0 & p_{23} & p_{31} & p_{12} \\ -p_{23} & 0 & p_{03} & -p_{02} \\ -p_{31} & -p_{03} & 0 & p_{01} \\ -p_{12} & p_{02} & -p_{01} & 0 \end{bmatrix}\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}=0 \qquad \text{(N)}$$

noting that the matrix has a zero determinant in view of (B), so that $X=x_j$ is not trivially zero.

Condition for a Plane to contain a Line

If u_j contains p_{jk} then the point X in (I) is singular i.e.

$$\begin{bmatrix} 0 & p_{01} & p_{02} & p_{03} \\ -p_{01} & 0 & p_{12} & -p_{31} \\ -p_{02} & -p_{12} & 0 & p_{23} \\ -p_{03} & p_{31} & -p_{23} & 0 \end{bmatrix}\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix}=0 \qquad \text{(R)}$$

A Pencil of Lines

If we are given two intersecting lines then the pencil is $p_{jk}=\lambda r_{jk}+\mu s_{jk}$. This follows because from (B) we have

$$0=(\lambda r_{01}+\mu s_{01})(\lambda r_{23}+\mu s_{23})+(\lambda r_{02}+\mu s_{02})(\lambda r_{31}+\mu s_{31})+(\lambda r_{03}+\mu s_{03})(\lambda r_{12}+\mu s_{12})$$
$$= \lambda^2(r_{01}r_{23}+ r_{02}r_{31}+r_{03}r_{12})+\mu^2(s_{01}s_{23}+ s_{02}s_{31}+s_{03}s_{12})+\lambda\mu(r_{01}s_{23}+r_{02}s_{31}+r_{03}s_{12}+r_{12}s_{03}+r_{31}s_{02}+r_{23}s_{01})$$

which is true by (B) and (C).

In Appendix A it is shown that the flat pencil centred on x_j in the plane u_j is

$$\begin{bmatrix} \lambda\{A+x_1B\}-\mu x_0(u_2-u_3) \\ \lambda\{A+x_2B\}-\mu x_0(u_3-u_1) \\ \lambda\{A+x_3B\}-\mu x_0(u_1-u_2) \\ \lambda u_0(x_1-x_2)-\mu\{A+u_3C\} \\ \lambda u_0(x_3-x_1)-\mu\{A+u_2C\} \\ \lambda u_0(x_2-x_3)-\mu\{A+u_1C\} \end{bmatrix} \qquad \text{(X)}$$

where $A = x_0u_0$,
$B = u_1+u_2+u_3$,
$C = x_1+x_2+x_3$.

There is a pencil of planes u_j in {1 1 1 0 0 0} for which (X) is degenerate, and for the line of points x_j on {0 0 0 1 1 1}.

The Regulus

Given three skew lines p_{jk} q_{jk} r_{jk} then there are ∞^1 instances of $\lambda p_{jk}+\mu q_{jk}+\nu r_{jk}$ which are lines, giving a regulus. This follows as (B) must be satisfied for a line, so

$$(\lambda p_{01}+\mu q_{01}+\nu r_{01})(\lambda p_{23}+\mu q_{23}+\nu r_{23})+(\lambda p_{02}+\mu q_{02}+\nu r_{02})(\lambda p_{31}+\mu q_{31}+\nu r_{31})+(\lambda p_{03}+\mu q_{03}+\nu r_{03})(\lambda p_{12}+\mu q_{12}+\nu r_{12})=0$$

Using (C1) the terms in λ^2, μ^2 and ν^2 are Ω_{pp}, Ω_{qq} and Ω_{rr} which are all zero by (B).
We are left with $\lambda\mu\Omega_{pq}$ $\lambda\nu\Omega_{pr}$ and $\mu\nu\Omega_{qr}$ which are not zero since the lines are skew, so the Plücker line condition is satisfied if

$$\lambda\mu\Omega_{pq} + \lambda\nu\Omega_{pr} + \mu\nu\Omega_{qr} = 0 \tag{Z}$$

for which there are ∞^1 distinct solutions for suitable values of λ μ and ν. Clearly p_{jk} belongs to this set of lines when μ and ν are zero, and similarly for q_{jk} and r_{jk}. From (I) the lines of $\lambda p_{jk}+\mu q_{jk}+\nu r_{jk}$ meet an arbitrary plane u_j in points with coordinates which are linear combinations of λ μ and ν, which describe a conic since (Z) is second order in λ μ and ν. Hence we have a regulus of lines containing p_{jk} q_{jk} and r_{jk}.

Another approach is to consider the lines s_{jk} which meet all three of p_{jk} q_{jk} and r_{jk}. From (C) we have

$$\begin{bmatrix} p_{01} & p_{02} & p_{03} & p_{12} & p_{31} & p_{23} \\ q_{01} & q_{02} & q_{03} & q_{12} & q_{31} & q_{23} \\ r_{01} & r_{02} & r_{03} & r_{12} & r_{31} & r_{23} \end{bmatrix} \begin{bmatrix} s_{23} \\ s_{31} \\ s_{12} \\ s_{03} \\ s_{02} \\ s_{01} \end{bmatrix} = 0$$

so partitioning the matrices as usual to solve the simultaneous homogeneous equations we have

$$\begin{bmatrix} \boldsymbol{P} & \boldsymbol{Q} \end{bmatrix} \begin{bmatrix} \boldsymbol{s}_{(23} \\ \boldsymbol{s}_{(03} \end{bmatrix} = 0$$

so

$$\boldsymbol{s}_{(23} = -\boldsymbol{P}^{-1}\boldsymbol{Q}\,\boldsymbol{s}_{(03}$$

and thus the three coordinates s_{01}, s_{02} and s_{03} of s_{jk} act as parameters for the other three. Two further constraints on s_{jk} are the line condition (B) and that they are homogeneous coordinates, leaving one degree of freedom and thus ∞^1 solutions for s_{jk}. They are not coplanar since p_{jk} q_{jk} and r_{jk} are skew so they form a regulus with p_{jk} q_{jk} and r_{jk} as directrices. This is the complementary regulus to that determined by (Z).

Transversals of Four Lines

Given four lines a_{jk} b_{jk} c_{jk} d_{jk} which are pairwise skew then in general there are two lines p_{jk} q_{jk} that intersect all four. This follows as a_{jk} b_{jk} c_{jk} determine a regulus $\mathcal{R}$, and d_{jk} meets it in at most two points through which pass two directrices of $\mathcal{R}$ meeting all four lines. From (C) the condition for such a transversal is

$$\begin{bmatrix} a_{01} & a_{02} & a_{03} & a_{12} & a_{31} & a_{23} \\ b_{01} & b_{02} & b_{03} & b_{12} & b_{31} & b_{23} \\ c_{01} & c_{02} & c_{03} & c_{12} & c_{31} & c_{23} \\ d_{01} & d_{02} & d_{03} & d_{12} & d_{31} & d_{23} \end{bmatrix} \begin{bmatrix} p_{01} \\ p_{02} \\ p_{03} \\ p_{12} \\ p_{31} \\ p_{23} \end{bmatrix} = 0$$

We partition this into

$$\begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \end{bmatrix} \begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{q} \end{bmatrix} = 0$$

so

$$\boldsymbol{p} = -\boldsymbol{A}^{-1} \boldsymbol{B} \boldsymbol{q}$$

where $\boldsymbol{A}$ is a non-singular 4x4 matrix obtained by permuting rows and columns if necessary, $\boldsymbol{B}$ is the remaining 2x4 matrix, and $\boldsymbol{p}$ and $\boldsymbol{q}$ are derived accordingly. Thus the four components of p_{jk} in $\boldsymbol{p}$ are determined by its two residual components in $\boldsymbol{q}$. Assuming $\boldsymbol{q}=\{p_{31}\ p_{23}\}$ we see that to satisfy (B) requires $p_{01}p_{23}+p_{01}p_{31}+p_{03}p_{12}=0$, and since the four components apart from p_{31} and p_{23} are linear functions of the latter we have a quadratic equation in p_{31}/p_{23} giving at most two real solutions.

Line at Infinity in a Plane

Given a plane $\{u_0\ u_1\ u_2\ u_3\}$, the coordinates of the line in which it meets the plane at infinity $\Gamma = \{0\ 0\ 0\ 1\}$ are $\pi_{jk}=\{0\ 0\ u_0\ 0\ {-u_1}\ u_2\}$ for the intersection of two planes, so by (O) the point-based coordinates of the line are $\{u_2\ \ {-u_1}\ 0\ u_0\ 0\ 0\}$.

Line at Infinity Conjugate to a given Line

Given a line p_{jk}, then by (J) the point a_j at infinity on the line is $\{p_{03}\ {-p_{31}}\ p_{23}\ 0\}$. The plane polar to a_j wrt Ψ is given by $\delta_{jk}a_k$ i.e. $\{a_0\ a_1\ a_2\ 0\}=\{p_{03}\ {-p_{31}}\ p_{23}\ 0\}$. This meets Γ in the line $\pi_{jk}=\{0\ 0\ p_{03}\ 0\ p_{31}\ p_{23}\}$ which by (O) is, for point coordinates, $q_{jk}=\{p_{23}\ p_{31}\ 0\ p_{03}\ 0\ 0\}$. This is the line in Γ conjugate to a_j.

Line of Intercept of a Plane with an invariant plane

By (A) a plane $\{a\ b\ c\ d\}$ meets the invariant plane $\alpha=\{1\ 0\ 0\ 0\}$ in the line $\pi=\{b\ c\ d\ 0\ 0\ 0\}$, or $p=\{0\ 0\ 0\ d\ c\ b\}$. The point $\{1\ 1\ 1\ 1\}$ does not lie in α, and if also it does not lie in $\{a\ b\ c\ d\}$ then by (V) p meets the invariant line $\{0\ 0\ 0\ 1\ 0\ 0\}$ in α in the point

$$\begin{bmatrix} & 0 & 0 & 0\\ 0 & & 1 & 0\\ 0 & -1 & & 0\\ 0 & 0 & 0 & \end{bmatrix}\begin{bmatrix} b+c+d\\ -b\\ -c\\ -d\end{bmatrix}=\begin{bmatrix}0\\ -c\\ b\\ 0\end{bmatrix}$$

and $\{0\ 0\ 0\ 0\ 1\ 0\}$ in α in

$$\begin{bmatrix} & 0 & 0 & 0\\ 0 & & 0 & -1\\ 0 & 0 & & 0\\ 0 & 1 & 0 & \end{bmatrix}\begin{bmatrix} b+c+d\\ -b\\ -c\\ -d\end{bmatrix}=\begin{bmatrix}0\\ d\\ 0\\ -b\end{bmatrix}$$

In the plane α the coordinates of these intercepts are $\{-c\ b\ 0\}$ and $\{d\ 0\ {-b}\}$ giving the planar equation of the line joining them as $\{-b^2\ \ {-bc}\ \ {-bd}\} \equiv \{b\ \ c\ \ d\}$ assuming $b\neq0$. If $b=0$ then using $\{0\ 0\ 0\ 0\ 0\ 1\}$ instead of $\{0\ 0\ 0\ 0\ 1\ 0\}$ we require $c\neq0$ i.e. we can choose so that the term cancelled $\neq0$, so the proviso is not needed.

Thus the plane $\{a\ b\ c\ d\}$ meets the plane $\{1\ 0\ 0\ 0\}$ in the line $\{b\ c\ d\ 0\ 0\ 0\} \equiv \{b\ \ c\ \ d\}$.

$\{a\ b\ c\ d\}$ meets $\beta=\{0\ 1\ 0\ 0\}$ in $\pi=\{a\ 0\ 0\ {-c}\ d\ 0\}$ or $p=\{0\ d\ {-c}\ 0\ 0\ a\}$. p meets the invariant line $\{0\ 1\ 0\ 0\ 0\ 0\}$ in β in

$$\begin{bmatrix} & 0 & 1 & 0\\ 0 & & 0 & 0\\ -1 & 0 & & 0\\ 0 & 0 & 0 & \end{bmatrix}\begin{bmatrix} a\\ -a-c-d\\ c\\ d\end{bmatrix}=\begin{bmatrix}c\\ 0\\ -a\\ 0\end{bmatrix}$$

and the line $\{0\ 0\ 0\ 0\ 0\ 1\}$ in

$$\begin{bmatrix} & 0 & 0 & 0\\ 0 & & 0 & 0\\ 0 & 0 & & 1\\ 0 & 0 & -1 & \end{bmatrix}\begin{bmatrix} a\\ -a-c-d\\ c\\ d\end{bmatrix}=\begin{bmatrix}0\\ 0\\ d\\ -c\end{bmatrix}$$

so in β the points are $\{c\ {-a}\ 0\}$ and $\{0\ d\ {-c}\}$ giving the line $\{ac\ c^2\ cd\} \equiv \{a\ c\ d\}$ (provided $c\neq0$).
Similarly for the plane $\{0\ 0\ 1\ 0\}$ we get $\pi=\{0\ a\ 0\ b\ 0\ {-d}\} \equiv \{a\ b\ d\}$

and for the plane {0 0 0 1}, π={0 0 a 0 -b c} ≡ {a b c}.
To summarise, the plane {a b c d} meets the invariant planes in the lines

{1 0 0 0} in π={b c d 0 0 0} ≡ {b c d} in line coordinates referred to the invariant plane,
{0 1 0 0} in π={a 0 0 -c d 0} ≡ {a c d} (AA)
{0 0 1 0} in π={0 a 0 b 0 -d} ≡ {a b d}
and {0 0 0 1} in π={0 0 a 0 -b c} ≡ {a b c}.

Perpendicular Projection of a Point onto a Plane

Let the plane be $U=u_j$ and the point be $X=x_j$. U intercepts Γ = {0 0 0 1} in the line $s_{jk}=\{u_2\ -u_1\ 0\ u_0\ 0\ 0\} \equiv \{u_0\ u_1\ u_2\}$ in plane coordinates by (AA). Let the point $Q=q_j$ in Γ be conjugate to s_{jk} wrt the absolute circle Ψ. The plane polar to Q wrt Ψ is given by $\delta_{jk}q_k$ i.e. $\{q_0\ q_1\ q_2\ 0\}$, which intersects Γ in $\{q_2\ -q_1\ 0\ q_0\ 0\ 0\}$ i.e. $q_j=u_j$ so that the point in Γ polar to the line where π meets it is simply $Q=\{u_0\ u_1\ u_2\ 0\}$. The line joining this to X is $p_{jk}=\{u_0x_1 - u_1x_0,\ u_0x_2 - u_2x_0,\ u_0x_3,\ u_1x_2 - u_2x_1,\ - u_1x_3,\ u_2x_3\}$ which by (I) intersects U in the point

$$\begin{bmatrix} 0 & u_0x_1-u_1x_0 & u_0x_2-u_2x_0 & u_0x_3-u_3x_0 \\ u_1x_0-u_0x_1 & 0 & u_1x_2-u_2x_1 & u_1x_3-u_3x_1 \\ u_2x_0-u_0x_2 & u_2x_1-u_1x_2 & 0 & u_2x_3-u_3x_2 \\ u_3x_0-u_0x_3 & u_3x_1-u_1x_3 & u_3x_2-u_2x_3 & 0 \end{bmatrix}\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix}$$
$$=\begin{bmatrix} u_0u_1x_1-u_1^2x_0+u_0u_2x_2-u_2^2x_0+u_0u_3x_3-u_3^2x_0 \\ u_0u_1x_0-u_0^2x_1+u_1u_2x_2-u_2^2x_1+u_1u_3x_3-u_3^2x_1 \\ u_0u_2x_0-u_0^2x_2+u_1u_2x_1-u_1^2x_2+u_2u_3x_3-u_3^2x_2 \\ u_0u_3x_0-u_0^2x_3+u_1u_3x_1-u_1^2x_3+u_2u_3x_2-u_2^2x_3 \end{bmatrix}$$
$$=\begin{bmatrix} u_0(u_1x_1+u_2x_2+u_3x_3)-x_0(u_1^2+u_2^2+u_3^2) \\ u_1(u_0x_0+u_2x_2+u_3x_3)-x_1(u_0^2+u_2^2+u_3^2) \\ u_2(u_0x_0+u_1x_1+u_3x_3)-x_2(u_0^2+u_1^2+u_3^2) \\ u_3(u_0x_0+u_1x_1+u_2x_2)-x_3(u_0^2+u_1^2+u_2^2) \end{bmatrix}$$

which may be expressed as

$$\{u_0X_0-x_0U_0,\ \ u_1X_1-x_1U_1,\ \ u_2X_2-x_2U_2,\ \ u_3X_3-x_3U_3\} \qquad \text{(A*)}$$

where

$$X_j=\Sigma u_i x_i \ - \ u_j x_j$$
$$U_j=\Sigma u_i^2 \ - \ u_j^2$$

Distance Between a Point and a Plane

Let the plane be $\pi=\{a\ b\ c\ d\}$. As above the pole of the line where π meets $\Gamma = \{0\ 0\ 0\ 1\}$ is $Q=\{a\ b\ c\ 0\}$. Now lines through Q are perpendicular to π, and thus the line though a point $P=\{x\ y\ z\ 1\}$ normal to π is the join of Q and P = $\{ay\text{-}bx, az\text{-}cx, a, bz\text{-}yc, \text{-}b, c\}$. By (I) this meets π in

$$Y=\begin{bmatrix} 0 & ay-bx & az-cx & a \\ bx-ay & 0 & bz-yc & b \\ cx-az & yc-bz & 0 & c \\ -a & -b & -c & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}=\begin{bmatrix} bay-b^2x+caz-c^2x+ad \\ abx-a^2y+cbz-yc^2+bd \\ acx-a^2z+byc-b^2z+cd \\ -a^2-b^2-c^2 \end{bmatrix}$$

Let $\lambda = \pi.P = ax+by+cz+d$. Then

$$Y=\begin{bmatrix} a(by+cz+d)-b^2x-c^2x \\ b(ax+cz+d)-a^2y-c^2y \\ c(ax+by+d)-a^2z-b^2z \\ -a^2-b^2-c^2 \end{bmatrix}=\begin{bmatrix} a(\lambda-ax)-b^2x-c^2x \\ b(\lambda-by)-a^2y-c^2y \\ c(\lambda-cz)-a^2z-b^2z \\ -a^2-b^2-c^2 \end{bmatrix}=\begin{bmatrix} a\lambda-(a^2+b^2+c^2)x \\ b\lambda-(a^2+b^2+c^2)y \\ c\lambda-(a^2+b^2+c^2)z \\ -a^2-b^2-c^2 \end{bmatrix}\equiv\begin{bmatrix} x-\frac{a}{s}\lambda \\ y-\frac{b}{s}\lambda \\ z-\frac{c}{s}\lambda \\ 1 \end{bmatrix}$$

where $s=a^2+b^2+c^2$.

The perpendicular distance between P and π is

$$\sqrt{\left(x-\frac{a}{s}\lambda-x\right)^2+\left(y-\frac{b}{s}\lambda-y\right)^2+\left(z-\frac{c}{s}\lambda-z\right)^2}=\sqrt{\frac{(a^2+b^2+c^2)\lambda^2}{s^2}}=\left|\frac{\lambda}{\sqrt{s}}\right|$$

Thus

$$\text{Perpendicular distance} = \left|\frac{ax+by+cz+d}{\sqrt{a^2+b^2+c^2}}\right|$$

which is the standard result but for homogeneous coordinates.

Distance Between a Point and a Line

Let the line be p_{jk} and the point be $X=x_{jk}$. Then the plane $U=u_{jk}$ common to them is given by (P):

$$U=\begin{bmatrix} 0 & p_{23} & p_{31} & p_{12} \\ -p_{23} & 0 & p_{03} & -p_{02} \\ -p_{31} & -p_{03} & 0 & p_{01} \\ -p_{12} & p_{02} & -p_{01} & 0 \end{bmatrix}\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ 1 \end{bmatrix}=\begin{bmatrix} p_{23}x_1+p_{31}x_2+p_{12} \\ -p_{23}x_0+p_{03}x_2-p_{02} \\ -p_{31}x_0-p_{03}x_1+p_{01} \\ -p_{12}x_0+p_{02}x_1-p_{01}x_2 \end{bmatrix}$$

This meets $\Gamma=\{0\ 0\ 0\ 1\}$ in the line $q_{jk}=\{u_0\ u_1\ u_2\}$ and the point z_{jk} in Γ conjugate to q_{jk} is $\{u_0\ u_1\ u_2\}$. Thus the plane π common to z_{jk} and p_{jk} is orthogonal to U, which by (P) is

$$\pi=\begin{bmatrix} 0 & p_{23} & p_{31} & p_{12} \\ -p_{23} & 0 & p_{03} & -p_{02} \\ -p_{31} & -p_{03} & 0 & p_{01} \\ -p_{12} & p_{02} & -p_{01} & 0 \end{bmatrix}\begin{bmatrix} p_{23}x_1+p_{31}x_2+p_{12} \\ -p_{23}x_0+p_{03}x_2-p_{02} \\ -p_{31}x_0-p_{03}x_1+p_{01} \\ 0 \end{bmatrix}$$

$$=\begin{bmatrix} -p_{23}^2x_0+p_{23}p_{03}x_2-p_{23}p_{02}-p_{31}^2x_0-p_{31}p_{03}x_1+p_{31}p_{01} \\ -p_{23}^2x_1-p_{23}p_{31}x_2-p_{23}p_{12}-p_{03}p_{31}x_0-p_{03}^2x_1+p_{03}p_{01} \\ -p_{31}p_{23}x_1-p_{31}^2x_2-p_{31}p_{12}+p_{03}p_{23}x_0-p_{03}^2x_2+p_{03}p_{02} \\ -p_{12}p_{23}x_1-p_{12}p_{31}x_2-p_{12}^2-p_{02}p_{23}x_0+p_{02}p_{03}x_2-p_{02}^2+p_{01}p_{31}x_0+p_{01}p_{03}x_1-p_{01}^2 \end{bmatrix}$$

giving the perpendicular distance from X to π, and therefore to p_{jk}, as

$$\left|\frac{\pi_0x_0+\pi_1x_1+\pi_2x_2+\pi_3}{\sqrt{\pi_0^2+\pi_1^2+\pi_2^2}}\right| \quad \text{assuming } x_3=1 \qquad \text{(B*)}$$

where $\pi_3 = (p_{01}p_{31}-p_{02}p_{32})x_0 + (p_{01}p_{03}-p_{12}p_{23})x_1 + (p_{02}p_{03}-p_{12}p_{31})x_2 - p_{01}^2 - p_{02}^2 - p_{12}^2$
and

$$\pi_0=-\left(p_{23}^2+p_{31}^2\right)x_0-p_{31}p_{03}x_1+p_{23}p_{03}x_2-p_{23}p_{02}+p_{31}p_{01}$$
$$\pi_1=-p_{03}p_{31}x_0-\left(p_{23}^2+p_{03}^2\right)x_1-p_{23}p_{31}x_2-p_{23}p_{12}+p_{03}p_{01}$$
$$\pi_2=p_{03}p_{23}x_0-p_{31}p_{23}x_1-\left(p_{31}^2+p_{03}^2\right)x_2-p_{31}p_{12}+p_{03}p_{02}$$

Note that the numerator $\pi_0x_0+\pi_1x_1+\pi_2x_2+\pi_3$ simplifies to

$$\begin{gathered} -\left(p_{23}x_0-p_{03}x_2\right)^2-\left(p_{31}x_0+p_{03}x_1\right)^2-\left(p_{23}x_1+p_{31}x_2\right)^2 \\ +2x_0\left(p_{31}p_{01}-p_{23}p_{02}\right) \\ +2x_1\left(p_{03}p_{01}-p_{23}p_{12}\right) \\ +2x_2\left(p_{03}p_{02}-p_{31}p_{12}\right) \\ -p_{12}^2-p_{02}^2-p_{01}^2 \end{gathered}$$

but for practical calculations it is easiest to evaluate the π_k individually as they are needed in the denominator.

<u>Angle Between a line and a Plane</u>

Let the plane be u_j and the line be p_{jk}. Their point of intersection $V=v_j$ is given by (I), and the perpendicular distance d from a point $X=x_j$ on p_{jk} is given by (B*). The points on p_{jk} are given by (M) enabling a suitable point X to be chosen. Then the angle is given by d and the length VX. Thus

$$\sin\theta=\frac{d}{\sqrt{\Sigma\left(v_j-x_j\right)^2}}$$

An alternative approach is to find the plane n_j in p_{jk} that intersects u_j in a line orthogonal to p_{jk}, and then find the angle between those planes. The plane u_j meets Γ in the line $m_j=\{u_0\ u_1\ u_2\ 0\}$ (c.f. (AA)) while p_{jk} meets it in the point $G=g_j=\{-p_{03}\ p_{31}\ -p_{23}\}$ (c.f. (D)). The planes n_j and u_j intersect in a line q_{jk} orthogonal to p_{jk}. Thus by (K) $p_{03}q_{03}+p_{31}q_{31}+p_{23}q_{23} = 0$. Now $p_{jk} = \{u_2n_3-u_3n_2, ...\}$, so $p_{03}(u_1n_2-u_2n_1)+p_{31}(u_0n_2-u_2n_0)+p_{23}(u_0n_1-u_1n_0)=0$

i.e. $0=\{-p_{31}u_2-p_{23}u_1,\ p_{23}u_0-p_{03}u_2,\ p_{03}u_1+p_{31}u_0\}.\{n_0\ n_1\ n_2\}=\sum w_j n_j$ say where w_j is a point lying in n_j. Also p_{jk} lies in n_j and thus by (I) $-p_{03}n_0+p_{31}n_1-p_{23}n_2=0=\sum n_j g_j$. Let z_j be the point in Γ polar to G lying in m_j and n_j, then $\sum z_j g_j=0$ and $\sum z_j m_j=0$. Also $\sum z_j n_j=0$. Thus for z_j:

$$\begin{bmatrix} g_0 & g_1 & g_2 \\ m_0 & m_1 & m_2 \end{bmatrix}\begin{bmatrix} z_0 \\ z_1 \\ z_2 \end{bmatrix}=0$$

$$\begin{bmatrix} g_0 & g_1 \\ m_0 & m_1 \end{bmatrix}\begin{bmatrix} z_0 \\ z_1 \end{bmatrix}=-z_2\begin{bmatrix} g_2 \\ m_2 \end{bmatrix}$$

i.e.

$$\begin{bmatrix} z_0 \\ z_1 \end{bmatrix}=-z_2\begin{bmatrix} m_1 & -g_1 \\ -m_0 & g_0 \end{bmatrix}\begin{bmatrix} g_2 \\ m_2 \end{bmatrix}=z_2\begin{bmatrix} m_2 g_1-m_1 g_2 \\ m_0 g_2-m_2 g_0 \end{bmatrix}$$

which solves z_j as ratios are sufficient. For n_j

$$\begin{bmatrix} g_0 & g_1 & g_2 \\ z_0 & z_1 & z_2 \end{bmatrix}\begin{bmatrix} n_0 \\ n_1 \\ n_2 \end{bmatrix}=0$$

$$\begin{bmatrix} g_0 & g_1 \\ z_0 & z_1 \end{bmatrix}\begin{bmatrix} n_0 \\ n_1 \end{bmatrix}=-n_2\begin{bmatrix} g_2 \\ z_2 \end{bmatrix}$$

i.e.

$$\begin{bmatrix} n_0 \\ n_1 \end{bmatrix}=-n_2\begin{bmatrix} z_1 & -g_1 \\ -z_0 & g_0 \end{bmatrix}\begin{bmatrix} g_2 \\ z_2 \end{bmatrix}=n_2\begin{bmatrix} g_1 z_2-g_2 z_1 \\ g_2 z_0-g_0 z_2 \end{bmatrix}$$

solves for n_j with z_j now known. Thus

$$\begin{bmatrix} n_0 \\ n_1 \end{bmatrix}=n_2\begin{bmatrix} g_1 z_2-g_2 z_2(m_0 g_2-m_2 g_0) \\ g_2 z_2(m_2 g_1-m_1 g_2)-g_0 z_2 \end{bmatrix}=n_2 z_2\begin{bmatrix} g_1-g_2(m_0 g_2-m_2 g_0) \\ g_2(m_2 g_1-m_1 g_2)-g_0 \end{bmatrix}$$

so setting $n_2 z_2=1$ this gives us n_j, and substituting p_{ij} for the g_j and u_j for the m_j gives

$$\begin{aligned} n_0 &= p_{31}-p_{23}^2 u_0-p_{23}p_{03}u_2 \\ n_1 &= p_{03}-p_{23}^2 u_1-p_{23}p_{31}u_2 \\ n_2 &= 1 \end{aligned}$$

A question remains as to whether n_2 can be zero. If it is then $n_0=n_1=0$ also and the plane is {0 0 0 1} i.e. the plane at infinity. Since n_j contains p_{jk} that would require the line to be at infinity, a special case for which there is no valid angle.

In three dimensions the required plane is $\{n_0\ n_1\ n_2\ 1\}$. Finally the angle between the planes u_j and n_j is

$$\cos\theta=\frac{u_0 n_0+u_1 n_1+u_2 n_2}{\sqrt{(u_0^2+u_1^2+u_2^2)(n_0^2+n_1^2+n_2^2)}} \qquad \text{(C*)}$$

which is the required angle between the line and plane. This assumes the line does not lie in the plane, and fails if so, so that must be checked first using (I). Neither must it lie in the plane at infinity. Thus we have a practically useful result after some manipulation which is simpler and more transparent than the first method.

Check: for the line OX={0 0 1 0 0 0} and the plane OZY={1 0 0 0}: n_j = {0 1 1} so cosθ=0 i.e. θ=90° which is correct. For OXY = {0 1 0 0} and OX, cosθ=1/√2 i.e. θ=45° which is wrong as expected.

The Line through a Point in a given Direction

Let the point be X=x_j in the direction v_j = {v_0 v_1 v_2}. The latter intersects Γ in I={v_0 v_1 v_2 0} so the required line is

$$\{x_0v_1-x_1v_0,\ x_0v_2-x_2v_0,\ -x_3v_0,\ x_1v_2-x_2v_1,\ x_3v_1,\ -\ x_3v_2\}$$

Angle between two Intersecting Lines

Let the lines be p_{jk} and q_{jk}. They meet Γ in (p_{03} -p_{31} p_{23} 0) and (q_{03} -q_{31} q_{23} 0) respectively, so their directions are given by the free vectors (p_{03} -p_{31} p_{23}) and (q_{03} -q_{31} q_{23}). The angle between them is given by their inner product, so the angle is

$$\theta=\operatorname{acos}\left(\frac{p_{03}q_{03}+p_{31}q_{31}+p_{23}q_{23}}{\sqrt{\left(p_{03}^2+p_{31}^2+p_{23}^2\right)\left(q_{03}^2+q_{31}^2+q_{23}^2\right)}}\right)$$

The Axis of a Null Polarity

Given a null polarity n_{jk} (where n_{jj}=0 and n_{jk}=-n_{kj}) the pole of the plane at infinity {0 0 0 1} is D=n_{k3} i.e. the last column of n_{jk}. The diameters contain this point. The polar line Q of D in Γ is q_k={n_{03} n_{13} n_{23}}, so a general plane in that line is u_j={n_{03} n_{13} n_{23} u_3}. The pole of such a plane wrt n_{jk} is $z_k=\Sigma n_{jk}u_j$ and thus the axis of n_{jk} is the join of D and z_k i.e. $p_{jk}=u_jz_k-u_kz_j = u_j\Sigma n_{sk}u_s - u_k\Sigma n_{sj}u_s$:

$$\begin{bmatrix} n_{03}(n_{01}n_{03}+n_{21}n_{23}+n_{31}u_3)-n_{13}(n_{10}n_{13}+n_{20}n_{23}+n_{30}u_3) \\ n_{03}(n_{02}n_{03}+n_{12}n_{13}+n_{32}u_3)-n_{23}(n_{10}n_{13}+n_{20}n_{23}+n_{30}u_3) \\ n_{03}(n_{03}n_{03}+n_{13}n_{13}+n_{23}n_{23})-u_3(n_{10}n_{13}+n_{20}n_{23}+n_{30}u_3) \\ n_{13}(n_{02}n_{03}+n_{12}n_{13}+n_{32}u_3)-n_{23}(n_{01}n_{03}+n_{21}n_{23}+n_{31}u_3) \\ u_3(n_{01}n_{03}+n_{21}n_{23}+n_{31}u_3)-n_{13}(n_{03}n_{03}+n_{13}n_{13}+n_{23}n_{23}) \\ n_{23}(n_{03}n_{03}+n_{13}n_{13}+n_{23}n_{23})-u_3 z_2 \end{bmatrix}$$

$$=\begin{bmatrix} n_{01}(n_{03}^2+n_{31}^2)-n_{23}(n_{03}n_{12}+n_{02}n_{31}) \\ n_{02}(n_{03}^2+n_{23}^2)-n_{31}(n_{03}n_{12}+n_{01}n_{23}) \\ n_{03}(n_{03}^2+n_{31}^2+n_{23}^2)-u_3 z_0 \\ n_{12}(n_{31}^2+n_{23}^2)-n_{03}(n_{01}n_{23}+n_{02}n_{31}) \\ n_{31}(n_{03}^2+n_{31}^2+n_{23}^2)+u_3 z_1 \\ n_{23}(n_{03}^2+n_{31}^2+n_{23}^2)-u_3 z_2 \end{bmatrix}$$

$$=\begin{bmatrix} n_{01}(n_{03}^2+n_{31}^2+n_{23}^2)-n_{23}(n_{01}n_{23}+n_{02}n_{31}+n_{03}n_{12}) \\ n_{02}(n_{03}^2+n_{31}^2+n_{23}^2)-n_{31}(n_{01}n_{23}+n_{02}n_{31}+n_{03}n_{12}) \\ n_{03}(n_{03}^2+n_{31}^2+n_{23}^2)-u_3 z_0 \\ n_{12}(n_{03}^2+n_{31}^2+n_{23}^2)-n_{03}(n_{01}n_{23}+n_{02}n_{31}+n_{03}n_{12}) \\ n_{31}(n_{03}^2+n_{31}^2+n_{23}^2)+u_3 z_1 \\ n_{23}(n_{03}^2+n_{31}^2+n_{23}^2)-u_3 z_2 \end{bmatrix}$$

Taking the plane in the origin with $u_3=0$ this gives $p_{jk}=Sn_{jk} - \Omega_n\{n_{23}\ n_{31}\ 0\ n_{03}\ 0\ 0\}$ defining S and Ω_n accordingly. If n_{jk} satisfies (B) and thus represents a line then $\Omega_n = 0$ giving $p_{jk} = n_{jk}$ as S may be cancelled provided it is not zero. If S is zero then p_{jk} is singular as the null-polarity is two-dimensional. Thus for a proper null polarity the axis is given by

$$S\{n_{01}\ n_{02}\ n_{03}\ n_{12}\ -n_{13}\ n_{23}\}-\Omega_n\{n_{23}\ -n_{13}\ 0\ n_{03}\ 0\ 0\} \qquad \text{(ANP)}$$

or

$$\{n_{01}\ n_{02}\ n_{03}\ n_{12}\ -n_{13}\ n_{23}\} \qquad \text{if } \Omega_n=0$$

where

$$S = n_{03}^2+n_{31}^2+n_{23}^2$$

and

$$\Omega_n = n_{01}n_{23}+n_{02}n_{31}+n_{03}n_{12}$$

Note that when $\Omega_n \neq 0$, S should not be omitted e.g. for the null polarity

$$\begin{bmatrix} 0 & d & 0 & 0 \\ -d & 0 & 0 & 0 \\ 0 & 0 & 0 & f \\ 0 & 0 & -f & 0 \end{bmatrix}$$

we have $f^2\{d\ 0\ 0\ 0\ 0\ f\}-fd\{f\ 0\ 0\ 0\ 0\ 0\} = \{0\ 0\ 0\ 0\ 0\ f\}$ i.e. the YO axis (as Y is taken as $\{0\ 0\ 1\ 0\}$ here).

The polar line of a given line wrt the null polarity is the intersection of the polar planes of two of its points.

Jessop (page 29) shows that in Klein coordinates the polar of a line z_j wrt a linear complex is $z_j+\lambda a_j$ where a_j are the coordinates of the complex and $\lambda=-2(az)/(a^2)$.
This transforms back to Pluecker coordinates as $q_{jk}=p_{jk}+\lambda a_{jk}$, where for evaluating λ we have $(az)=\{a_{01}z_{23}+a_{02}z_{31}+a_{03}z_{12}+a_{12}z_{03}+a_{31}z_{02}+a_{23}z_{01}\}=\Omega_{az}$ in Pluecker coordinates,
and $(a^2)=-2\{a_{01}a_{23}+a_{02}a_{31}+a_{03}a_{12}\}=-\Omega_{aa}=-2\Omega_a$, so $\lambda=-2\Omega_{az}/\Omega_{aa}$ Thus in our notation

$$\rho q_{jk}= \Omega_{nn}p_{jk}-2\Omega_{pn}n_{jk} \qquad \text{(PNP)}$$

Thus the axis, when it belongs to the complex ($\Omega_{pn}=0$), is polar to itself (definition of a null line) while otherwise it is $q_{jk} =\{(S+\lambda)n_{01}+\Omega_n n_{23}, (S+\lambda)n_{02}+\Omega_n n_{31}, (S+\lambda)n_{03}, (S+\lambda)n_{12}+\Omega_n n_{03}, -(S+\lambda)n_{13}, (S+\lambda)n_{23}\}$.
Now $\Omega_{nn} = 2\{n_{01}n_{23}+n_{02}n_{31}+n_{03}n_{12}\} = 2\Omega_n$ and
$\Omega_{qn} = \Omega_{nn}(S+\lambda)+\Omega_n(n_{31}{}^2+n_{23}{}^2+n_{03}{}^2) = \Omega_{nn}S-2\Omega_{qn}+S\Omega_{nn}/2$ so $\Omega_{qn} = S\Omega_{nn}/2$
and finally $\lambda = -2\Omega_{qn}/\Omega_{nn} = -S$.
Thus the polar line q_{jk} of the axis is $\{n_{23}, n_{31}, 0, n_{03}, 0, 0\}$ i.e. the line at infinity as shown previously.

Summary

The coordinates of a line given by two planes are related to those given by two points by

$$\pi_{01} : \pi_{02} : \pi_{03} : \pi_{12} : \pi_{31} : \pi_{23} = p_{23} : p_{31} : p_{12} : p_{03} : p_{02} : p_{01} \tag{O}$$

The Plücker condition for the p_{jk} to represent a line is

$$p_{01}p_{23} + p_{02}p_{31} + p_{03}p_{12} = 0 \tag{B}$$

The condition for two lines to intersect is

$$p_{01}q_{23}+p_{02}q_{31}+p_{03}q_{12}+p_{12}q_{03}+p_{31}q_{02}+p_{23}q_{01} = 0 \tag{C}$$

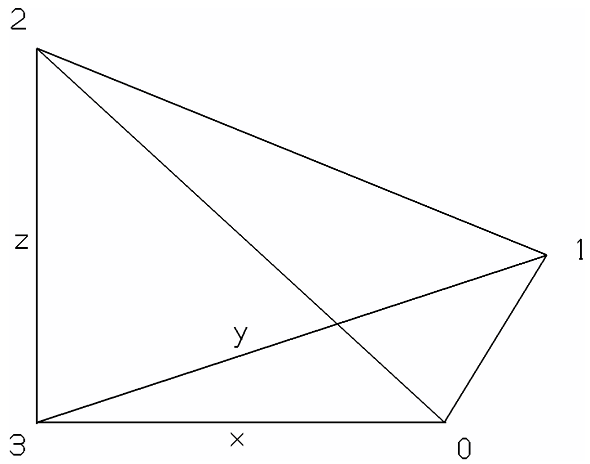


Figure 1

The x-axis OX is the line joining vertices 3 and 0, which has all coordinates zero except p_{03}, etc..
The line in a plane has those coordinates zero which contain the number of the opposite vertex.
The line meeting a vertex has those coordinates non-zero which contain the number for that vertex.
The line meeting an edge has one coordinate zero given by the numbers of the opposite (skew) edge.

A line meets the plane at infinity in the point

$$\{-p_{03}\ p_{31}\ -p_{23}\ 0\} \tag{D}$$

The point X where a line meets a plane u_j is

$$X = \begin{bmatrix} 0 & p_{01} & p_{02} & p_{03} \\ -p_{01} & 0 & p_{12} & -p_{31} \\ -p_{02} & -p_{12} & 0 & p_{23} \\ -p_{03} & p_{31} & -p_{23} & 0 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{I}$$

The plane U determined by a line and a point is

$$U=\begin{bmatrix} 0 & p_{23} & p_{31} & p_{12} \\ -p_{23} & 0 & p_{03} & -p_{02} \\ -p_{31} & -p_{03} & 0 & p_{01} \\ -p_{12} & p_{02} & -p_{01} & 0 \end{bmatrix}\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{P}$$

Two lines are skew-orthogonal if

$$p_{03}q_{03}+p_{31}q_{31}+p_{23}q_{23} = 0 \tag{K}$$

A variable point on a line is

$$x_j = \{\lambda p_{03} , -\lambda p_{31}-\mu p_{01} , \lambda p_{23}-\mu p_{02} , -\mu p_{03}\} \tag{M}$$

A variable plane in a line is:

$$u_j = \{\lambda p_{12} , -\lambda p_{02}-\mu p_{23} , \lambda p_{01}-\mu p_{31} , -\mu p_{12}\} \tag{Q}$$

A point lies on a line if

$$\begin{bmatrix} 0 & p_{23} & p_{31} & p_{12} \\ -p_{23} & 0 & p_{03} & -p_{02} \\ -p_{31} & -p_{03} & 0 & p_{01} \\ -p_{12} & p_{02} & -p_{01} & 0 \end{bmatrix}\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}=0 \tag{N}$$

A plane contains a line if:

$$\begin{bmatrix} 0 & p_{01} & p_{02} & p_{03} \\ -p_{01} & 0 & p_{12} & -p_{31} \\ -p_{02} & -p_{12} & 0 & p_{23} \\ -p_{03} & p_{31} & -p_{23} & 0 \end{bmatrix}\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix}=0 \tag{R}$$

The point of intersection of two intersecting lines is

$$y_j=\begin{bmatrix} & q_{01} & q_{02} & q_{03} \\ -q_{01} & & q_{12} & -q_{31} \\ -q_{02} & -q_{12} & & q_{23} \\ -q_{03} & q_{31} & -q_{23} & \end{bmatrix}\begin{bmatrix} p_{23}+p_{31}+p_{12} \\ -p_{23}+p_{03}-p_{02} \\ -p_{31}-p_{03}+p_{01} \\ -p_{12}+p_{02}-p_{01} \end{bmatrix} \tag{V}$$

The plane common to two intersecting lines is

$$u_j=\begin{bmatrix} & q_{23} & q_{31} & q_{12} \\ -q_{23} & & q_{03} & -q_{02} \\ -q_{31} & -q_{03} & & q_{01} \\ -q_{12} & q_{02} & -q_{01} & \end{bmatrix}\begin{bmatrix} p_{01}+p_{02}+p_{03} \\ -p_{01}+p_{12}-p_{31} \\ -p_{02}-p_{12}+p_{23} \\ -p_{03}+p_{31}-p_{23} \end{bmatrix} \tag{W}$$

The line in the plane at infinity into which p_{jk} is projected from the origin is

$$\{p_{12}\ -p_{02}\ p_{01}\} \tag{S}$$

A line {u v w} in the plane at infinity has the Plücker coordinates

$$\{w\ -v\ 0\ u\ 0\ 0\} \tag{T}$$

The flat pencil centred on x_j in the plane u_j is

$$\begin{bmatrix} \lambda\left(A+x_1 B\right)-\mu\, x_0\left(u_2-u_3\right) \\ \lambda\left(A+x_2 B\right)-\mu\, x_0\left(u_3-u_1\right) \\ \lambda\left(A+x_3 B\right)-\mu\, x_0\left(u_1-u_2\right) \\ \lambda\, u_0\left(x_1-x_2\right)-\mu\left(A+u_3 C\right) \\ \lambda\, u_0\left(x_3-x_1\right)-\mu\left(A+u_2 C\right) \\ \lambda\, u_0\left(x_2-x_3\right)-\mu\left(A+u_1 C\right) \end{bmatrix} \tag{X}$$

where $A = x_0 u_0$,
$B = u_1+u_2+u_3$,
$C = x_1+x_2+x_3$.

The plane {a b c d} meets the invariant planes

{1 0 0 0} in π={b c d 0 0 0} ≡ {b c d} in coordinates referred to the invariant plane,
{0 1 0 0} in π={a 0 0 -c d 0} ≡ {a c d}
{0 0 1 0} in π={0 a 0 b 0 -d} ≡ {a b d}
and {0 0 0 1} in π={0 0 a 0 -b c} ≡ {a b c}.

The perpendicular distance between a plane {a b c d} and a point {x y z 1} is

$$\left|\frac{ax+by+cz+d}{\sqrt{a^2+b^2+c^2}}\right|$$

The axis of a null polarity n_{jk} is $\{n_{01}\ n_{02}\ n_{03}\ n_{12}\ -n_{13}\ n_{23}\}$.

Annex A

Pencil of Lines

We seek the pencil of lines p_{jk} in a plane u_j with a point x_j as centre in that plane (so $\Sigma x_j u_j=0$).
We find the points A and B where u_j meets the lines {1 0 0 0 0 0} and {0 0 0 0 0 1} respectively, and then the lines a_{ij} and b_{ij} joining x_j to A and B respectively. Then the pencil is $\lambda a_{ij}+\mu b_{ij}$.
Using (I)

$$A=\begin{bmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}u_0\\u_1\\u_2\\u_3\end{bmatrix}=\begin{bmatrix}u_1\\-u_0\\0\\0\end{bmatrix}$$

and

$$B=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&-1&0\end{bmatrix}\begin{bmatrix}u_0\\u_1\\u_2\\u_3\end{bmatrix}=\begin{bmatrix}0\\0\\u_3\\-u_2\end{bmatrix}$$

so

$$a_{ij}=\{x_0u_0+x_1u_1,\ \ x_2u_1,\ \ x_3u_1,\ \ -x_2u_0,\ \ x_3u_0,\ \ 0\}$$
$$b_{ij}=\{0,\ \ -x_0u_3,\ \ x_0u_2,\ \ -x_1u_3,\ \ -x_1u_2,\ \ x_2u_2+x_3u_3\}$$

giving the pencil (using (A))

$$\begin{bmatrix}\lambda(x_0u_0+x_1u_1)\\ \lambda x_2u_1-\mu x_0u_3\\ \lambda x_3u_1+\mu x_0u_2\\ -\lambda x_2u_0-\mu x_1u_3\\ \lambda x_3u_0-\mu x_1u_2\\ \mu(x_2u_2+x_3u_3)\end{bmatrix} \qquad \text{(A1)}$$

If either {1 0 0 0 0 0} or {0 0 0 0 0 1} lies in the plane u_j then either a_{ij} or b_{ij} will be singular, so another pair of skew lines of the tetrahedron must be chosen instead.
The pencil based on {0 1 0 0 0 0} and {0 0 0 0 1 0} is similarly:

$$\begin{bmatrix}\lambda x_1u_2+\mu x_0u_3\\ \lambda(x_0u_0+x_2u_2)\\ \lambda x_3u_2-\mu x_0u_1\\ \lambda x_1u_0-\mu x_2u_3\\ \mu(x_1u_1+x_3u_3)\\ -\lambda x_3u_0-\mu x_2u_1\end{bmatrix} \qquad \text{(A2)}$$

and that on {0 0 1 0 0 0} and {0 0 0 1 0 0} is

$$\begin{bmatrix} \lambda x_1 u_3 - \mu x_0 u_2 \\ \lambda x_2 u_3 + \mu x_0 u_1 \\ \lambda (x_0 u_0 + x_3 u_3) \\ \mu (x_1 u_1 + x_2 u_2) \\ -\lambda x_1 u_0 - \mu x_3 u_2 \\ \lambda x_2 u_0 - \mu x_3 u_1 \end{bmatrix} \tag{A3}$$

Now three lines of a pencil may be added to give another line of that pencil, so we may add the λ terms in (A1) (A2) and (A3) to give a line which cannot be singular. That is because u_j cannot contain all three of {1 0 0 0 0 0} {0 1 0 0 0 0} and {0 0 1 0 0 0} because they meet in the vertex 0 of Figure 1 and so are not coplanar. Similarly the μ terms may be added, giving us the pencil with no danger of singularity:

$$\begin{bmatrix} \lambda\{x_0 u_0 + x_1(u_1+u_2+u_3)\} - \mu x_0 (u_2 - u_3) \\ \lambda\{x_0 u_0 + x_2(u_1+u_2+u_3)\} - \mu x_0 (u_3 - u_1) \\ \lambda\{x_0 u_0 + x_3(u_1+u_2+u_3)\} - \mu x_0 (u_1 - u_2) \\ \lambda u_0 (x_1 - x_2) - \mu \{x_0 u_0 + u_3(x_1+x_2+x_3)\} \\ \lambda u_0 (x_3 - x_1) - \mu \{x_0 u_0 + u_2(x_1+x_2+x_3)\} \\ \lambda u_0 (x_2 - x_3) - \mu \{x_0 u_0 + u_1(x_1+x_2+x_3)\} \end{bmatrix} \tag{A4}$$

where the identity $\Sigma x_j u_j = 0$ has been exploited to replace e.g. $x_1u_1+x_2u_2$ by $-x_0u_0-x_3u_3$. We may then express this as

$$\begin{bmatrix} \lambda\{A + x_1 B\} - \mu x_0 (u_2 - u_3) \\ \lambda\{A + x_2 B\} - \mu x_0 (u_3 - u_1) \\ \lambda\{A + x_3 B\} - \mu x_0 (u_1 - u_2) \\ \lambda u_0 (x_1 - x_2) - \mu \{A + u_3 C\} \\ \lambda u_0 (x_3 - x_1) - \mu \{A + u_2 C\} \\ \lambda u_0 (x_2 - x_3) - \mu \{A + u_1 C\} \end{bmatrix} \tag{A5}$$

where $A = x_0 u_0$,
$B = u_1+u_2+u_3$,
$C = x_1+x_2+x_3$.

If e.g. $u_0=0$ and $B=0$ then the pencil degenerates to a single line. u_j then contains the line {1 1 1 0 0 0} joining the unit point {1 1 1 1} to {1 0 0 0}. Similarly if $x_0=C=0$, x_j then lying in the line of intersection of the unit plane and {1 0 0 0} i.e. {0 0 0 1 1 1}. Since $\{0, x_1, x_2, -x_1-x_2\}$ does not lie in $\{0, u_1, u_2, -u_1-u_2\}$ both conditions cannot occur simultaneously, so (A5) is never singular (of course λ and μ must not both be zero). Thus there is a pencil of planes in {1 1 1 0 0 0} for which (A5) is degenerate, and the line of points {0 0 0 1 1 1}.