

HOW SPACE AND COUNTERSPACE INTERACT TO PRODUCE PATH CURVES

by

N.C.THOMAS

The work of Lawrence Edwards (Ref 2) has produced impressive evidence that certain fundamental projective forms accurately describe plant buds, eggs, seed pods, and the heart. The work is based on that of George Adams who recognised the likely application of path curves in this way. George Adams developed the theory of counterspace, or polar Euclidean space, to correspond with Rudolf Steiner's description of negative space. Euclidean space is characterised in particular by the ability to determine the right angle and to make measurements that are invariant with respect to change of position or direction. The projective geometric characterisation of these properties is secured by confining the allowable linear space transformations to those which leave the plane at infinity invariant, together with an imaginary circle in that plane known as the Absolute Imaginary Circle (Ref 5). The latter is a degenerate form of quadric, which it is important to note since the general projective theory of non-Euclidean spaces requires some quadric to be defined as the Absolute. George Adams investigated the case when an imaginary cone is taken as that Absolute. This is the polar of the Absolute Imaginary Circle of Euclidean space, and in place of an invariant plane at infinity each such space has a "point at infinity", namely the real vertex of the imaginary cone. Hence such a space may be referred to as a polar Euclidean space (Ref 1). In it planes act as fundamental entities analogously to the role of points in Euclidean space. A point, on the other hand, appears as a compound entity composed of the stars of planes and lines it contains (Ref 1 gives a detailed exposition).

This conception of negative space lies behind the research carried out by George Adams and Olive Whicher (Ref 1) as well as by Lawrence Edwards. In the study of path curves (see below) it is important to find the relationship of the process to counter-space. Lawrence Edwards describes his work mostly in terms of point-wise collineations of space, for the sake of brevity and clarity. This does not reveal, however, the way in which space and counter-space interact to produce path curves. George Adams states that this is an unsolved problem (Ref 1 note 41). A solution and its consequences will now be described.

If a space collineation is repeatedly applied to a point then successive positions of that point will describe an invariant curve of the transformation known as a path curve. A plane repeatedly transformed in this way will envelope a developable surface of which the cuspidal edge is a path curve. Generally such a transformation leaves the four points, four planes and six edges of a tetrahedron invariant (Ref 2). In the case of collineations yielding egg and vortex forms two of the invariant planes are parallel and the other two are imaginary, while two of the invariant points are the Absolute Circling Points at infinity of the real invariant planes (Ref 2).

Now, a collineation may be taken as the product of two correlations. A correlation transforms points into planes and vice versa (Ref 3, where it is referred to as a reciprocity). This possibility allows for a "breathing" or interaction to take place between space and counter-space. If we start with some point P , then a correlation will transform it into some plane π . A second different correlation will then transform π back into a point P' distinct from P . The product of these two correlations has the same effect as a collineation transforming P into P' . In this way we may regard P and P' as members of Euclidean space, and π as a member of counter-space.

How do we determine these correlations in a significant way, considering the infinite possibilities open to us? Jessop shows (Ref 3) that a general correlation is determined by a quadric and a linear complex. In view of the relationship between the linear complex and forces, both physical and universal (Ref 4), this seems a significant approach to adopt. The simplest and most obvious case is to let the correlation from space to counter-space be

determined by the counter-space Absolute Imaginary Cone together with some linear complex, and to let the transformation back from counter-space to space be determined by the Euclidean Absolute together with the same linear complex, but now considered from a plane-wise point of view. The two ways of regarding a linear complex, either as relating to physical forces, or as the result of considering all its polar complexes in relation to universal forces, is described in detail by George Adams (Ref 4).

Using the language of matrices we find the second correlation is expressed as

$$\begin{bmatrix} 1 & f & 0 & 0 \\ -f & 1 & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & -d & 0 \end{bmatrix} \quad (3)$$

where f and d are constants determining a complex with its axis coincident with the z-axis (see Annex 2; equation numbers refer to those in Annex 2). Then $-f/d=k$, the fundamental constant of the complex (Ref 3 paragraph 23).

If we place the vertex of the counter-space Absolute on the z-axis, then the matrix for the first correlation is

$$= \begin{bmatrix} 1 & d & 0 & 0 \\ -d & 1 & 0 & 0 \\ 0 & 0 & 1 & f-c \\ 0 & 0 & -(f+c) & c^2 \end{bmatrix} \quad (2)$$

where f and d are as for equation (3), and c is the Euclidean z-coordinate of the vertex of the counter-space Absolute (see Annex 2 for the derivation of this and all following results).

The product of these two correlations yields the matrix for the collineation:-

$$= \begin{bmatrix} 1-fd & f+d & 0 & 0 \\ -(f+d) & 1-fd & 0 & 0 \\ 0 & 0 & 1-d(f+c) & f-c+dc^2 \\ 0 & 0 & -d & -d(f-c) \end{bmatrix} \quad (4)$$

This matrix now determines the path curve system. Notice that the only assumption made is to place the vertex of the imaginary cone on the z-axis, apart from the adoption of Jessop's method of determining the correlations in conjunction with a linear complex and the Absolutes of space and counter-space.

The collineation (4) possesses two real and two imaginary double points, the latter being the Absolute Circling Points at infinity in its two real invariant planes (cf Ref 2). This is precisely what we want if we are to describe the work of Lawrence Edwards from the perspective adopted here.

The coordinates of the double points are:

$$\begin{aligned} Q_1 : & \quad (-i, 1, 0, 0) \\ Q_2 : & \quad (i, 1, 0, 0) \\ Q_3 : & \quad (0, 0, dc - 0.5 + \{0.25 - fd\}, d) \\ Q_4 : & \quad (0, 0, dc - 0.5 - \{0.25 - fd\}, d) \end{aligned} \quad (13)$$

It immediately follows that the height of the bud (or vortex) is:

$$H = \frac{\sqrt{1-4fd}}{d} \quad (11)$$

and that the relative height of the vertex of the counter-space Absolute is:

$$\frac{h}{H} = \frac{1}{1 \pm e^{\mp \varepsilon \theta}}$$

where ε is the parameter defined by Lawrence Edwards which determines the spiraling of the path curves and θ is the angular step (Ref 2). This relative height is the ratio of the distance of the vertex from Q_4 to the height given by (11). For buds the etheric centre (as we shall call the vertex of the counter-space Absolute) lies on the axis inside the bud, between Q_3 and Q_4 . For vortices it may lie outside Q_3 and Q_4 , below the vortex, or between Q_3 and Q_4 (see Fig 4). Thus two types of vortex are obtainable, distinguished by the position of the etheric centre.

The parameter ε is given by :

$$\varepsilon = \frac{\tanh^{-1}(1-4fd)^{\pm \frac{1}{2}}}{\theta}$$

and

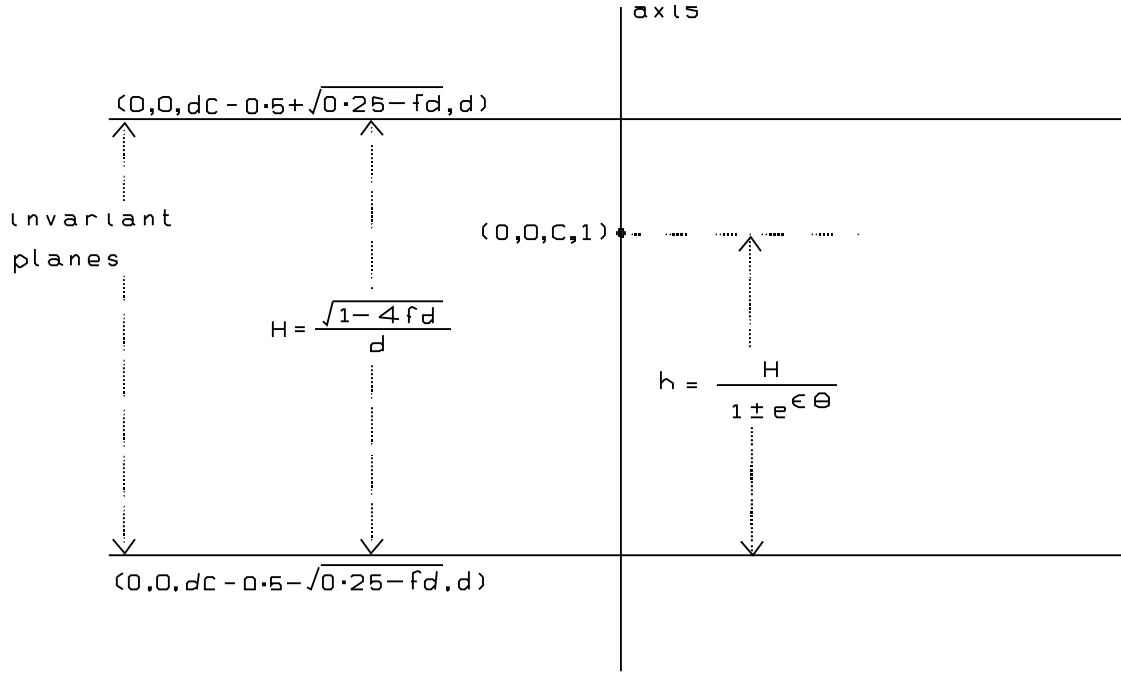
$$\theta = -\tan^{-1}\left(\frac{f+d}{1-fd}\right) \quad (7)$$

Lawrence Edwards' parameter λ is given by :

$$\lambda = -\frac{\log \sqrt{(1+f^2)(1+d^2)} - 2\log \left[\frac{1}{2} - \sqrt{\frac{1}{4} - fd} \right]}{\log \sqrt{(1+f^2)(1+d^2)} - 2\log \left[\frac{1}{2} + \sqrt{\frac{1}{4} - fd} \right]} \quad (6)$$

The remarkable thing to notice is that the parameter c , defining the location of the etheric centre, is entirely absent in these equations: the parameters of the linear complex completely determine λ , ε , θ , h and H . Thus wherever we place the etheric centre, the path curve system will form itself in relation to that centre as determined by the above equations.

The following diagram, showing the axis and real invariant planes, summarises our findings so far :



Equations (9) and (6) may be solved to yield d and f in terms of ϵ , λ and θ . We find that :

$$d^2 = \frac{\varphi \pm \sqrt{\varphi^2 - 4[(e^{\epsilon \theta} \pm 1)(1 \pm e^{-\epsilon \theta})]^{-2}}}{2}$$

$$f^2 = \frac{\varphi \mp \sqrt{\varphi^2 - 4[(e^{\epsilon \theta} \pm 1)(1 \pm e^{-\epsilon \theta})]^{-2}}}{2} \quad (20)$$

where

$$\varphi = \left[(e^{\epsilon \theta} \pm 1)(1 \pm e^{-\epsilon \theta})^\lambda \right]^{-\frac{4}{1+\lambda}} - 1 - [(e^{\epsilon \theta} \pm 1)(1 \pm e^{-\epsilon \theta})]^{-2}$$

The sign ambiguities in the exponentials are not easy to resolve. Sometimes there are two solutions, sometimes only one. Figure 3 shows plots of λ and ϵ on $f.d$ axes. The solution of transcendental equations is required to find f and d from λ and ϵ , although they may be found from ϵ and θ directly. Generally, if the product of f and d is positive then the path curves are vortical, but if that product is negative then either buds or vortices are found depending on the values of f and d . The position of the etheric centre follows the sign of $f.d$ i.e. if negative then it lies between the invariant planes, but if positive then outside them.

Although f and d are independently selectable, not all combinations yield the case of the semi-imaginary invariant tetrahedron we have been considering. We find that if $f.d > 0.25$ then λ and ϵ become imaginary (cf. (6) and (9)), and all double points become imaginary (see equations (13)). The same process also yields real path

curves in the fully imaginary case, for the collineation (4) remains real.

Equations (20) show that d and f need not be real for all combinations of θ and $\epsilon\theta$. We find that for f and d to be real,

$$\lambda \geq -\frac{2 \log P + \log[1+1PQ]}{2 \log Q + \log[1+1PQ]} \quad (23)$$

$$\begin{aligned} \text{where } P &= e^{\epsilon\theta} \pm 1 \\ \text{and } Q &= 1 \pm e^{-\epsilon\theta} \end{aligned}$$

(the signs must be consistent with equations (9) - (11)).

This equation is summarised in Table 1 and Figure 1, where the shaded portions show the areas for which f^2 and d^2 are imaginary. The "asymptotic locus" is a sketch of the locus of points determining asymptotic path curves i.e. path curves which are the asymptotic lines of their vortical invariant surfaces. It will not fail to be noticed that f^2 and d^2 are imaginary for such cases.

These results show that for buds all combinations of λ and ϵ are obtainable by suitable choice of f and d , but not in the case of vortices.

So far the "strong" theory has been described: strong in the sense that it requires the fewest parameters. If instead of using the same linear complex to determine both correlations (2) and (3) we use two distinct coaxial complexes then similar results are obtained. However, no restrictions are then placed on λ and ϵ for vortices, allowing in particular asymptotic curves to be path curves. In addition to pure rotations, biaxial collineations and space homologies arise as special cases when two or more eigenvalues coincide. Thus pure rotations are obtainable in all cases, and both axial and central expansions in the more general theory (N.B. the biaxial collineation has for its path curves the lines of a linear congruence).

Even more general correlations are possible by allowing the complexes to have distinct axes. However, general expressions for the eigenvalues - and hence for λ , ϵ and θ - are more difficult to obtain since the characteristic equation of the transformation is not readily factorised. Such cases are easily studied numerically, but general insight into the relationships involved is not thereby gained.

CONCLUSIONS

A collineation may be regarded as the product of two correlations. In this way the interworking of space and counter-space may be expressed. The simplest way to define correlations, involving both the Euclidean and polar Euclidean Absolutes - together with a linear complex - immediately yields a collineation of the required type. That is: one with a semi-imaginary tetrahedron and possessing axial symmetry.

A more complicated solution using two linear complexes yields similar results but with more parameters and fewer restrictions.

Thus a path curve system may indeed be regarded as arising from a relationship between space and counter-space, and the implications may be worked out in detail. The process postulated is a rhythmic interchange between physical and etheric substance in the case of living organisms, the etherialisation of physical nourishment being expressed by the correlation from space to counter-space, and the resulting organic growth or

repair by the reverse correlation. The parameter θ would seem to express this rhythm rather than an actual spatial rotation.

In the application to water vortices the process postulated is a tendency to crystallize, expressed by the correlation from counter-space to space, balanced by a tendency to evaporate as expressed by the reverse correlation. The fluid state may be envisaged as an equilibrium between these two tendencies.

REFERENCES

1. "The Plant Between Sun and Earth", George Adams and Olive Whicher, Rudolf Steiner Press 1980.
2. "The Field of Form", Lawrence Edwards, Floris Books 1982.
3. "A Treatise on the Line Complex", C.M.Jessop, C.U.P. 1903.
4. "Universal Forces in Mechanics", George Adams, Rudolf Steiner Press 1977.
5. "Algebraic Projective Geometry", Semple and Kneebone, O.U.P. 1952.

ANNEXES

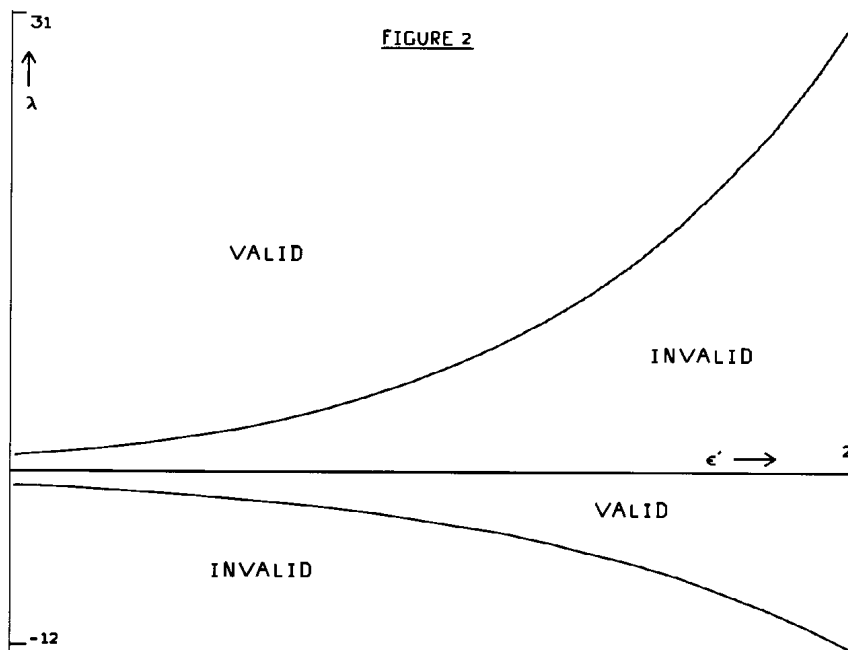
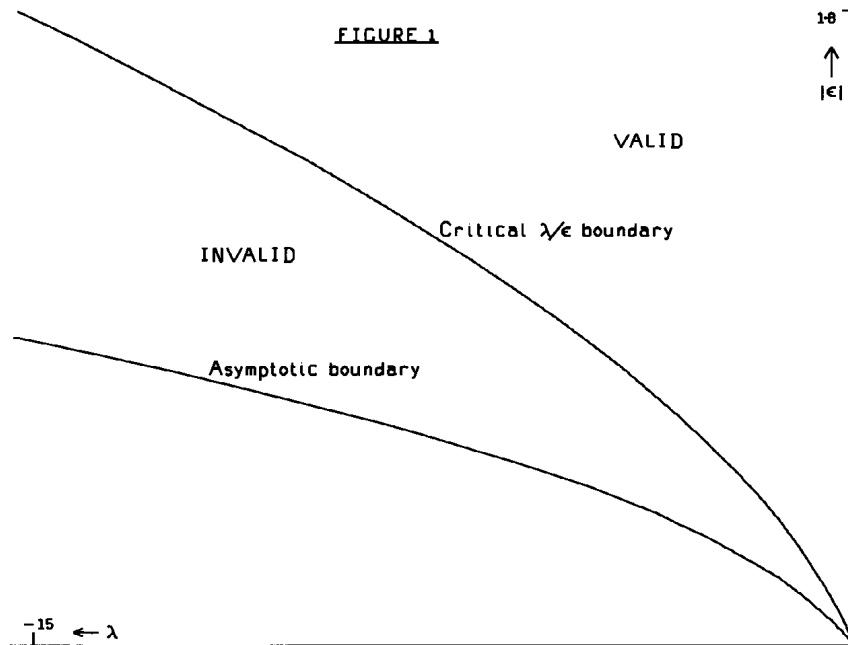
1. General Analytic Theory.
2. Particular Solutions.

FIGURES

1. Relation of critical λ and ϵ for vortices, with asymptotic locus.
2. Relation of f and d for real λ and ϵ .
3. λ and ϵ loci on f,d graph.
4. Position of Etheric Centre.

TABLES

1. Solutions of Quartic Equation Determining Sign of λ .



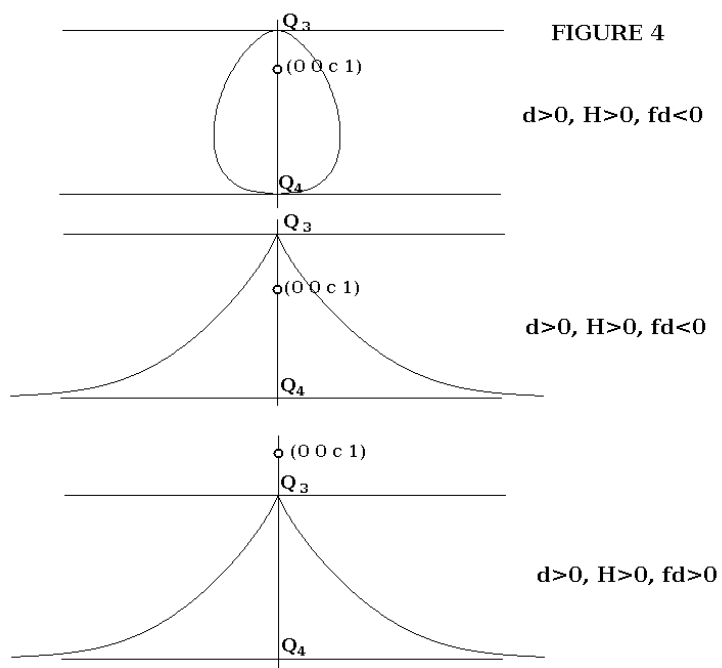
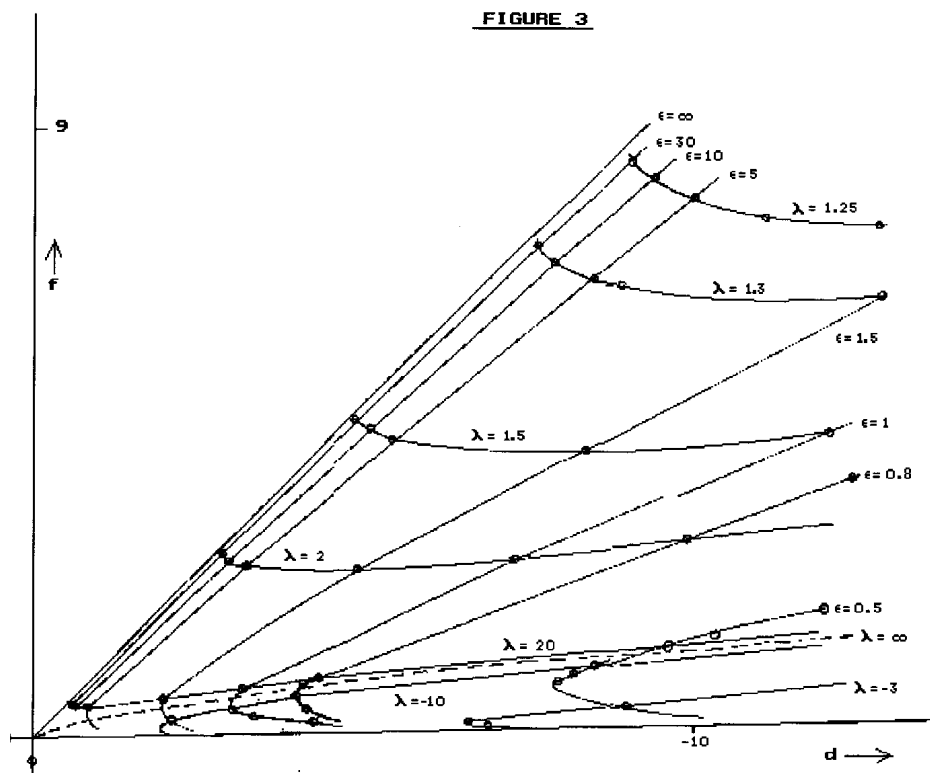


TABLE 1

Solutions of quartic equation determining sign of λ

d	e	d	e
0.001	-0.003732	0.01	-0.037325
0.001	-0.000268	0.01	-0.002679
0.002	-0.007464	0.02	-0.074673
0.002	-0.000536	0.02	-0.005359
0.003	-0.011196	0.03	-0.112070
0.003	-0.000804	0.03	-0.149540
0.004	-0.014928	0.04	-0.008038
0.004	-0.001072	0.04	-0.010717
0.005	-0.018661	0.05	-0.187106
0.005	-0.001340	0.05	-0.013395
0.006	-0.022393	0.06	-0.224794
0.006	-0.001608	0.06	-0.016072
0.007	-0.026126	0.07	-0.262629
0.007	-0.001876	0.07	-0.018749
0.008	-0.029858	0.08	-0.300635
0.008	-0.002144	0.08	-0.021425
0.009	-0.033591	0.09	-0.338837
0.009	-0.002412	0.09	-0.024100
0.1	-0.377261	1.0	-7.870932
0.1	-0.026774	1.0	-0.250463
0.2	-0.779401	2.0	-39.899496
0.2	-0.053425	2.0	-0.439968
0.3	-1.232360	3.0	-119.924906
0.3	-0.079833	3.0	-0.582024
0.4	-1.761634	4.0	-271.941151
0.4	-0.105888	4.0	-0.695146
0.5	-2.391577	5.0	-519.951914
0.5	-0.131491	5.0	-0.789611
0.6	-3.145780	6.0	-887.959456
0.6	-0.156560	6.0	-0.871150
0.7	-4.047526	7.0	-1399.964997
0.7	-0.181029	7.0	-0.943211
0.8	-5.120062	8.0	-2079.969227
0.8	-0.204853	8.0	-1.008014
0.9	-6.386716	9.0	-2951.972559
0.9	-0.228002	9.0	-1.067067

ANNEX 1

GENERAL ANALYTIC THEORY

1. The following general results are used to facilitate the derivation of special solutions. A transformation of the following general type is considered:

$$\begin{bmatrix} \alpha & \beta & 0 & 0 \\ \gamma & \delta & 0 & 0 \\ 0 & 0 & \sigma & \rho \\ 0 & 0 & \mu & \tau \end{bmatrix}$$

To find the eigenvalues, which we require to determine the invariant points of the transformation, we must solve the determinant

$$\begin{vmatrix} \alpha - \Lambda_i & \beta & 0 & 0 \\ \gamma & \delta - \Lambda_i & 0 & 0 \\ 0 & 0 & \sigma - \Lambda_i & \rho \\ 0 & 0 & \mu & \tau - \Lambda_i \end{vmatrix} = 0$$

where the Λ_i are the eigenvalues. The characteristic equation is:

$$[(\alpha - \Lambda_i)(\delta - \Lambda_i) - \beta\gamma][(\sigma - \Lambda_i)(\tau - \Lambda_i) - \mu\rho] = 0$$

$$\text{i.e. } [\Lambda_i^2 - (\alpha + \delta)\Lambda_i + \alpha\delta - \beta\gamma][\Lambda_i^2 - (\sigma + \tau)\Lambda_i + \sigma\tau - \mu\rho] = 0$$

Hence

$$\Lambda_i = \frac{1}{2}[\alpha + \delta \pm \sqrt{(\alpha - \delta)^2 + 4\beta\gamma}] \quad (1)$$

or

$$\Lambda_i = \frac{1}{2}[\sigma + \tau \pm \sqrt{(\sigma - \tau)^2 + 4\mu\rho}] \quad (2)$$

2. The eigenvectors are found by solving the equations :

$$\begin{bmatrix} \alpha - \Lambda_i & \beta & 0 & 0 \\ \gamma & \delta - \Lambda_i & 0 & 0 \\ 0 & 0 & \sigma - \Lambda_i & \rho \\ 0 & 0 & \mu & \tau - \Lambda_i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = 0$$

Dividing by w , as we are using homogeneous coordinates, we may solve the first three equations to find the ratios x/w , y/w and z/w :

$$\begin{bmatrix} \alpha - \Lambda_i & \beta \\ \gamma & \delta - \Lambda_i \\ & \sigma - \Lambda_i \end{bmatrix} \begin{bmatrix} x/w \\ y/w \\ z/w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\rho \end{bmatrix}$$

By Cramer's rule we have :

$$\frac{x}{w} : \frac{y}{w} : \frac{z}{w} = \frac{\begin{vmatrix} 0 & \beta & 0 \\ 0 & \delta - \Lambda_i & 0 \\ -\rho & 0 & \sigma - \Lambda_i \end{vmatrix}}{\Delta} : \frac{\begin{vmatrix} \alpha - \Lambda_i & 0 & 0 \\ \gamma & 0 & 0 \\ 0 & -\rho & \sigma - \Lambda_i \end{vmatrix}}{\Delta} : \frac{\begin{vmatrix} \alpha - \Lambda_i & \beta & 0 \\ \gamma & \delta - \Lambda_i & 0 \\ 0 & 0 & -\rho \end{vmatrix}}{\Delta}$$

where Δ is the determinant

$$\begin{vmatrix} \alpha - \Lambda_i & \beta & 0 \\ \gamma & \delta - \Lambda_i & 0 \\ 0 & 0 & \sigma - \Lambda_i \end{vmatrix}$$

This gives, after cancellation of δ ,

$$x : y : z : w = (0 : 0 : -\rho : \sigma - \Lambda_i)$$

so the two eigenvectors are

$$(0, 0, \rho, \Lambda_i - \sigma) \tag{4}$$

Similarly, solving the last three equations of (3) by Cramer's rule, we get

$$(\Lambda_i - \delta, \gamma, 0, 0) \tag{5}$$

Substituting from (2) in (4) we get

$$(0, 0, 2\rho, \tau - \sigma \pm \sqrt{(\sigma - \tau)^2 + 4\mu\rho}) \quad (6)$$

and substitution from (1) in (5) gives

$$(\alpha - \delta \pm \sqrt{(\alpha - \delta)^2 + 4\beta\gamma}, 2\gamma, 0, 0) \quad (7)$$

For a semi-imaginary tetrahedron we require two of the eigenvectors to be the absolute circling points at infinity. Equations (7) are evidently our candidates as they represent points on a line at infinity. These must be

$$(\pm i, 1, 0, 0)$$

$$\text{so } \alpha = \delta \text{ and } \beta = -\gamma \quad (8)$$

Hence we have the transformation

$$\begin{bmatrix} \alpha & \beta & 0 & 0 \\ -\beta & \alpha & 0 & 0 \\ 0 & 0 & \sigma & \rho \\ 0 & 0 & \mu & \tau \end{bmatrix} \quad (9)$$

and from (6) the "height", i.e. the distance between the real double points on the z-axis, is

$$\frac{2\rho}{\tau - \sigma - \sqrt{(\tau - \sigma)^2 + 4\mu\rho}} - \frac{2\rho}{\tau - \sigma + \sqrt{(\tau - \sigma)^2 + 4\mu\rho}} = \frac{\sqrt{(\tau - \sigma)^2 + 4\mu\rho}}{-\mu} \quad (10)$$

provided $\rho > 0$. The upper real invariant point Q_3 is taken to correspond to Λ_3 with the negative sign in (2), and Λ_1 and Λ_2 are taken as the imaginary double points.

3. Now we wish to find the parameters λ , ϵ and θ of the path curve transformation (ref (2) page 61).

We transform the general Euclidean point (x,y,z,1)

$$\begin{bmatrix} \alpha & \beta & 0 & 0 \\ -\beta & \alpha & 0 & 0 \\ 0 & 0 & \sigma & \rho \\ 0 & 0 & \mu & \tau \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\alpha x + \beta y}{\eta} \\ -\frac{\beta x + \alpha y}{\eta} \\ \frac{\sigma z + \rho}{\eta} \\ 1 \end{bmatrix}$$

$$\text{where } \eta = \mu z + \tau$$

Suppose (x, y, z, 1) lies in a real invariant plane. Then we require

$$z = \frac{\sigma z + \rho}{\eta}$$

so substituting for η and solving for z gives

$$z = \frac{\sigma - \tau \pm \sqrt{(\sigma - \tau)^2 + 4\mu\rho}}{2\mu}$$

but, from (6) we know that

$$z = \frac{2\rho}{\tau - \sigma \pm \sqrt{(\sigma - \tau)^2 + 4\mu\rho}}$$

which gives the identity

$$z = \frac{\sigma - \tau \pm \sqrt{(\sigma - \tau)^2 + 4\mu\rho}}{2\mu} = \frac{2\rho}{\tau - \sigma \pm \sqrt{(\sigma - \tau)^2 + 4\mu\rho}} \quad (11)$$

which is easily verified by cross multiplying.

It also gives an alternative expression for the real invariant points

$$(0, 0, \sigma - \tau \pm \sqrt{(\sigma - \tau)^2 + 4\mu\rho}, 2\mu) \quad (6a)$$

also

$$\eta + \mu z = \frac{\sigma - \tau \pm \sqrt{(\sigma - \tau)^2 + 4\mu\rho}}{2} + \tau = \frac{\sigma + \tau \pm \sqrt{(\sigma - \tau)^2 + 4\mu\rho}}{2}$$

which by equation (2) is Λ_3 or Λ_4 .

Before the transformation the radius of the point $(x, y, z, 1)$ in the invariant plane is

$$\sqrt{x^2 + y^2}$$

After transformation its radius is

$$\frac{\sqrt{(\alpha x + \beta y)^2 + (\alpha y - \beta x)^2}}{\eta} = \frac{\sqrt{\alpha^2 + \beta^2} \sqrt{x^2 + y^2}}{\eta}$$

Hence the ratio of the radii is $\frac{\sqrt{\alpha^2 + \beta^2}}{\eta}$

To decide which Λ_i to associate with which invariant plane in place of η , note that a projective transformation of the point (x, y, z, w) using the diagonal form of the transform is

$$(\Lambda_1 x, \Lambda_2 y, \Lambda_3 z, \Lambda_4 w)$$

and for a point in a real invariant plane we require

$$\frac{\Lambda_3 z}{\Lambda_4 w} = \frac{z}{w} \quad \text{i.e. } (\Lambda_3 - \Lambda_4)zw = 0$$

so $z=0$ or $w=0$ for unequal Λ_i . If $z=0$ the Euclidean point in the upper invariant plane is

$$\left(\frac{\Lambda_1 x}{\Lambda_4 w}, \frac{\Lambda_2 y}{\Lambda_4 w}, 0, 1 \right)$$

and for the lower invariant plane we have

$$\left(\frac{\Lambda_1 x}{\Lambda_3 w}, \frac{\Lambda_2 y}{\Lambda_3 w}, 0, 1 \right)$$

Hence in the top invariant plane the ratio is $\frac{\sqrt{\alpha^2 + \beta^2}}{\Lambda_4}$

and in the lower it is $\frac{\sqrt{\alpha^2 + \beta^2}}{\Lambda_3}$

giving λ as the negative ratio of logs (ref (2)) :

$$\lambda = - \frac{\log \sqrt{\alpha^2 + \beta^2} - \log \Lambda_4}{\log \sqrt{\alpha^2 + \beta^2} - \log \Lambda_3} \quad (12)$$

4. Clearly the angle θ turned through is given by

$$\theta = -\tan^{-1}(\beta/\alpha) \quad (13)$$

5. To find ε we recall that

$$\lambda = \frac{\varepsilon + \alpha'}{\varepsilon - \alpha'} \quad (\text{ref (2) page 61})$$

using α' to distinguish it from our α , and comparing with (12) we find that

$$\alpha' + \varepsilon' = \log \sqrt{\alpha^2 + \beta^2} - \log \Lambda_4$$

and

$$\alpha' - \varepsilon' = \log \sqrt{\alpha^2 + \beta^2} - \log A_3$$

ε' has been used since the value derived from these equations will be per θ radians turned, whereas the true ε is defined per unit radian.

Hence $\varepsilon' = 0.5 \log (\Lambda_3/\Lambda_4)$ (14)

and $\varepsilon = \frac{\varepsilon'}{\theta} = -\frac{\log(A_3/A_4)}{2 \tan^{-1}(\beta/\alpha)}$ (15)

An alternative expression for ε' is

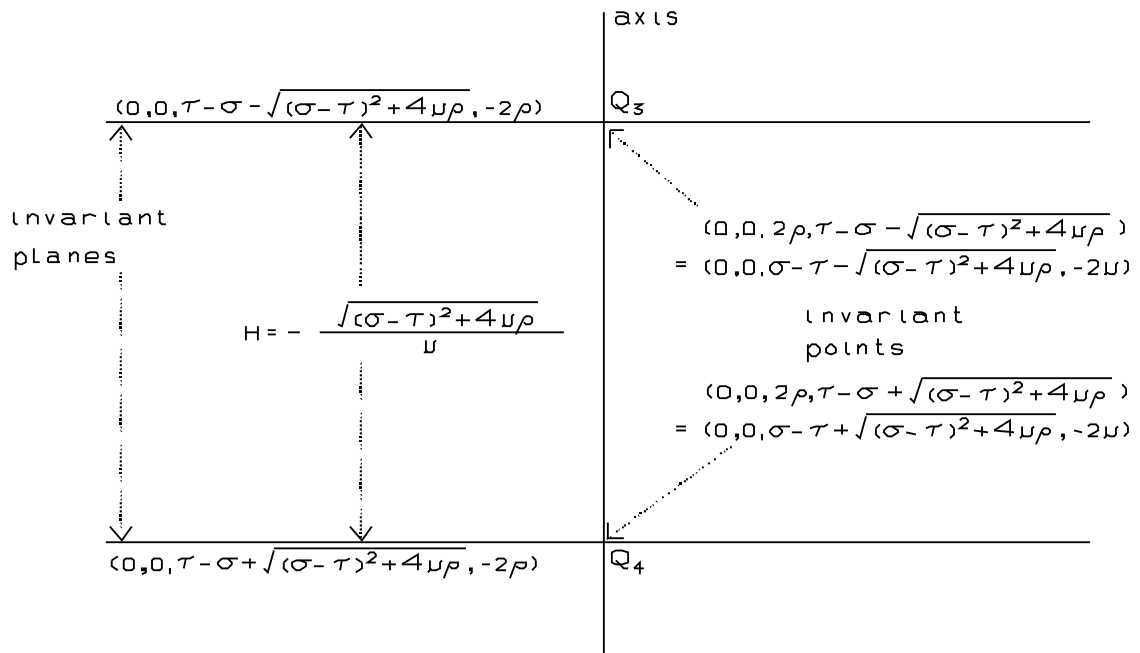
$$\begin{aligned} \varepsilon' &= 0.5 \log \left[\frac{\sigma + \tau + \sqrt{(\sigma - \tau)^2 + 4\mu\rho}}{\sigma + \tau - \sqrt{(\sigma - \tau)^2 + 4\mu\rho}} \right] \text{ from (2)} \\ &= \tanh^{-1} \left[\frac{\sqrt{(\sigma - \tau)^2 + 4\mu\rho}}{\sigma + \tau} \right] \\ \tanh \varepsilon' &= \frac{\sqrt{(\sigma - \tau)^2 + 4\mu\rho}}{\sigma + \tau} \quad (16) \end{aligned}$$

6. We note that

$$(\sigma - \tau)^2 + 4\mu\rho \geq 0 \quad (17)$$

for the real double points to be obtained as required.

7. The above results are summarised in the following diagram:



8. The transformation referred to the invariant tetrahedron is

$$\begin{bmatrix} \varphi e^{i\theta} & 0 & 0 & 0 \\ 0 & \varphi e^{-i\theta} & 0 & 0 \\ 0 & 0 & A_3 & 0 \\ 0 & 0 & 0 & A_4 \end{bmatrix} \quad (18)$$

$$= \begin{bmatrix} \varphi^{\frac{1}{\theta}} e^i & 0 & 0 & 0 \\ 0 & \varphi^{\frac{1}{\theta}} e^{-i} & 0 & 0 \\ 0 & 0 & A_3^{\frac{1}{\theta}} & 0 \\ 0 & 0 & 0 & A_4^{\frac{1}{\theta}} \end{bmatrix}^{\theta} \quad (19)$$

It is useful to express the same transformation referred to plane coordinates. Let the collineation be $\mathbf{A}$, so that

$$\mathbf{x}' = \mathbf{A} \mathbf{x} \quad \text{for the transformation of a point } \mathbf{x}.$$

Let the planewise collineation equivalent to $\mathbf{A}$ be $\mathbf{B}$. Then

$$\mathbf{B} = (\mathbf{A}^{-1})\mathbf{T} \quad (\text{ref 5 page 348}).$$

Applying this to (18) gives

$$\begin{bmatrix} \left(\frac{1}{\varphi}\right) e^{-i\theta} & 0 & 0 & 0 \\ 0 & \left(\frac{1}{\varphi}\right) e^{i\theta} & 0 & 0 \\ 0 & 0 & \frac{1}{A_3} & 0 \\ 0 & 0 & 0 & \frac{1}{A_4} \end{bmatrix}$$

which is equivalent to

$$\begin{bmatrix} \varphi e^{-i\theta} & 0 & 0 & 0 \\ 0 & \varphi e^{i\theta} & 0 & 0 \\ 0 & 0 & \frac{\varphi^2}{A_3} & 0 \\ 0 & 0 & 0 & \frac{\varphi^2}{A_4} \end{bmatrix} = \begin{bmatrix} \varphi e^{-i\theta} & 0 & 0 & 0 \\ 0 & \varphi e^{i\theta} & 0 & 0 \\ 0 & 0 & A_3' & 0 \\ 0 & 0 & 0 & A_4' \end{bmatrix} \quad (20)$$

Now

$$\begin{aligned} \lambda &= -\frac{\log \varphi - \log A_3}{\log \varphi - \log A_4} = \frac{\log \varphi + \log(\varphi^2 / A_3) - 2 \log \varphi}{\log \varphi + \log(\varphi^2 / A_4) - 2 \log \varphi} \\ &= -\frac{\log \varphi - \log A_3'}{\log \varphi - \log A_4'} \end{aligned} \quad (21)$$

i.e. the formula for λ is identical for a planewise collineation.

$$\varepsilon' = 0.5 \log \frac{A_3}{A_4} = 0.5 \log \frac{A_4'}{A_3'} \quad (22)$$

But

$$\varepsilon = \frac{\varepsilon'}{\theta} = \frac{0.5 \log(A_4' / A_3')}{\theta} = \frac{0.5 \log(A_3' / A_4')}{-\theta} \quad (23)$$

Hence if (20) is re-arranged to give θ the same sign as the pointwise collineation i.e.

$$\begin{bmatrix} \varphi e^{i\theta} & 0 & 0 & 0 \\ 0 & \varphi e^{-i\theta} & 0 & 0 \\ 0 & 0 & A_3' & 0 \\ 0 & 0 & 0 & A_4' \end{bmatrix} \quad (24)$$

then the formulae for λ and ε are identical, apart from ε' being reversed in sign.

ANNEX 2

PARTICULAR SOLUTIONS

1. The equations derived in this Annex were found first of all. However, as alternative possibilities came to be studied a more general solution became desirable. This is presented in Annex 1, which eases the work now to be described.

2. To determine the correlations required, it is first necessary to specify the matrices of the Euclidean and Counterspace Absolutes. This requires care as both are degenerate imaginary quadrics. We will first consider the imaginary cone forming the Counterspace Absolute. Jessop shows (Ref 3 para 50) that the most general correlation (he refers to it as a reciprocity) is determined by a quadric a_{ik} and a linear complex A_{ik} (expressed here as a null polarity) so that the correlation is

$$B_{ik} = a_{ik} + A_{ik}.$$

Thus we require first the matrix a_{ik} of the quadric. An imaginary cone may be described by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

if it is treated pointwise. This gives an imaginary circular cone with its vertex at the point $Q_4 = (0, 0, 0, 1)$, the origin of the Euclidean system as described in homogeneous coordinates. If we now shift the origin along the z-axis, the vertex has the equation $(0, 0, c, 1)$ where c is a parameter, and the equation of the cone becomes $x^2 + y^2 + (z - cw)^2 = 0$, where (x, y, z, w) is a general point. Hence the matrix we require is

$$a_{ik} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -c \\ 0 & 0 & -c & c^2 \end{bmatrix} \quad (1)$$

3. We now need the matrix of a suitable linear complex described in terms of its null system. Jessop shows that a linear complex with the z-axis for its axis is of the form

$$A_{ik} = \begin{bmatrix} 0 & d & 0 & 0 \\ -d & 0 & 0 & 0 \\ 0 & 0 & 0 & f \\ 0 & 0 & -f & 0 \end{bmatrix} \quad (1A)$$

(ref 3 page 31 where $d = a_{12}$ and $f = a_{34}$).

These are point-wise matrices suitable for defining a correlation from point space to plane space, or Euclidean to Counter space. Hence the correlation is

$$B_{ik} = a_{ik} + A_{ik} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -c \\ 0 & 0 & -c & c^2 \end{bmatrix} + \begin{bmatrix} 0 & d & 0 & 0 \\ -d & 0 & 0 & 0 \\ 0 & 0 & 0 & f \\ 0 & 0 & -f & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & d & 0 & 0 \\ -d & 1 & 0 & 0 \\ 0 & 0 & 1 & f-c \\ 0 & 0 & -(f+c) & c^2 \end{bmatrix} \quad (2)$$

Note that we have assumed that the vertex of the cone lies on the z-axis. This corresponds to the assumption that the fourth coordinate line of our system of axes is in the plane at infinity containing the Euclidean Absolute. These are polar assumptions.

4. For the return correlation from Counter space we need to develop an approach polar to that of Jessop i.e. to describe the Euclidean Absolute and the linear complex in plane coordinates. This is fortunate as a matrix may be found for the Euclidean Absolute in plane coordinates, and is simply

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

for any plane π such that

$$[\pi_1 \pi_2 \pi_3 \pi_4] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{bmatrix} = 0$$

(i.e. $\pi_1^2 + \pi_2^2 + \pi_3^2 = 0$) touches the absolute imaginary circle, for the point $(\pi_1, \pi_2, \pi_3, 0)$ lies in the plane, and also satisfies the equations

$$x^2 + y^2 + z^2 = 0, \quad w = 0 \quad \text{of the absolute circle.}$$

5. Finally we need to describe the linear complex in plane coordinates. On page 18 of Ref 3 Jessop shows the relationship between point- and plane- based line coordinates :

$$p_{12} : p_{23} : p_{31} : p_{14} : p_{24} : p_{34} = \pi_{34} : \pi_{14} : \pi_{24} : \pi_{23} : \pi_{31} : \pi_{12}$$

A linear complex is given by the equations

$$\sum_{i,k} a_{ik} p_{ik} = 0$$

where the a_{ik} are the six constants determining the complex and the p_{ik} are the Plücker coordinates of any line belonging to it. For plane based coordinates we have

$$\sum_{r,s} b_{rs} \pi_{rs} = 0$$

Hence from the relationship between the p_{ik} and the π_{rs} we deduce one between the a_{ik} and the b_{rs} , namely r,s are the complement of i,k in the set (1 2 3 4), in the usual order for Plücker coordinates.

Thus, noting that the indices for a_{ik} are valid for both the matrix and Plücker elements, the plane-wise matrix for the linear complex is from (1A):

$$\begin{bmatrix} 0 & f & 0 & 0 \\ -f & 0 & 0 & 0 \\ 0 & 0 & 0 & d \\ 0 & 0 & -d & 0 \end{bmatrix}$$

and the required correlation from Counter to Euclidean space is:

$$\begin{aligned} C_{ij} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & f & 0 & 0 \\ -f & 0 & 0 & 0 \\ 0 & 0 & 0 & d \\ 0 & 0 & -d & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & f & 0 & 0 \\ -f & 1 & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & -d & 0 \end{bmatrix} \end{aligned} \quad (3)$$

6. Since we will consider a pointwise collineation, it will be formed by the product of (2) and (3), transforming a point by (2) first and then the resulting plane by (3). Hence our collineation is:

$$\begin{aligned} G_{ij} &= C_{ik} B_{kj} = \begin{bmatrix} 1 & f & 0 & 0 \\ -f & 1 & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & -d & 0 \end{bmatrix} \begin{bmatrix} 1 & d & 0 & 0 \\ -d & 1 & 0 & 0 \\ 0 & 0 & 1 & f-c \\ 0 & 0 & -(f+c) & c^2 \end{bmatrix} \\ &= \begin{bmatrix} 1-fd & f+d & 0 & 0 \\ -(f+d) & 1-fd & 0 & 0 \\ 0 & 0 & 1-d(f+c) & f-c+dc^2 \\ 0 & 0 & -d & -d(f-c) \end{bmatrix} \end{aligned} \quad (4)$$

Comparing with (9) in Annex 1 we see that provided f , c and d satisfy (17) of Annex 1, this collineation has a semi-imaginary tetrahedron of the required type, where

$$\begin{aligned} \alpha &= 1 - fd \\ \beta &= f + d \\ \sigma &= 1 - d(f + c) \\ \rho &= f - c + dc^2 \\ \mu &= -d \\ \tau &= d(c - f) \end{aligned} \quad (5)$$

Hence by implementing the most natural choices for determining a collineation from two correlations, namely:

1. One quadric is the Euclidean Absolute,
2. The other is the Counter space Absolute,
3. The same linear complex is used for both correlations,
4. The vertex of the Counter space Absolute lies on the axis of the complex

we obtain immediately a path curve transformation of the type studied by Lawrence Edwards.

7. From equations (12), (13), (14) and (15) of Annex 1, and using (5) above, we see that

$$\lambda = - \frac{\log \sqrt{(1+f^2)(1+d^2)} - \log \left(\frac{1}{2} - \sqrt{\frac{1}{4} - fd} \right)^2}{\log \sqrt{(1+f^2)(1+d^2)} - \log \left(\frac{1}{2} + \sqrt{\frac{1}{4} - fd} \right)^2} \quad (6)$$

$$\theta = -\tan^{-1} \left(\frac{f+d}{1-fd} \right) \quad (7)$$

noting that $1 - 2fd \pm \sqrt{1 - 4fd} = (1 \pm \sqrt{1 - 4fd})^2 / 2$

$$\varepsilon' = \frac{1}{2} \log \left(\frac{1 + \sqrt{1 - 4fd}}{1 - \sqrt{1 - 4fd}} \right)^2 \quad (8)$$

$$\varepsilon = \frac{\log \left(\frac{1 + \sqrt{1 - 4fd}}{1 - \sqrt{1 - 4fd}} \right)^2}{2 \tan^{-1} \left(\frac{f+d}{1-fd} \right)} \quad (9)$$

so

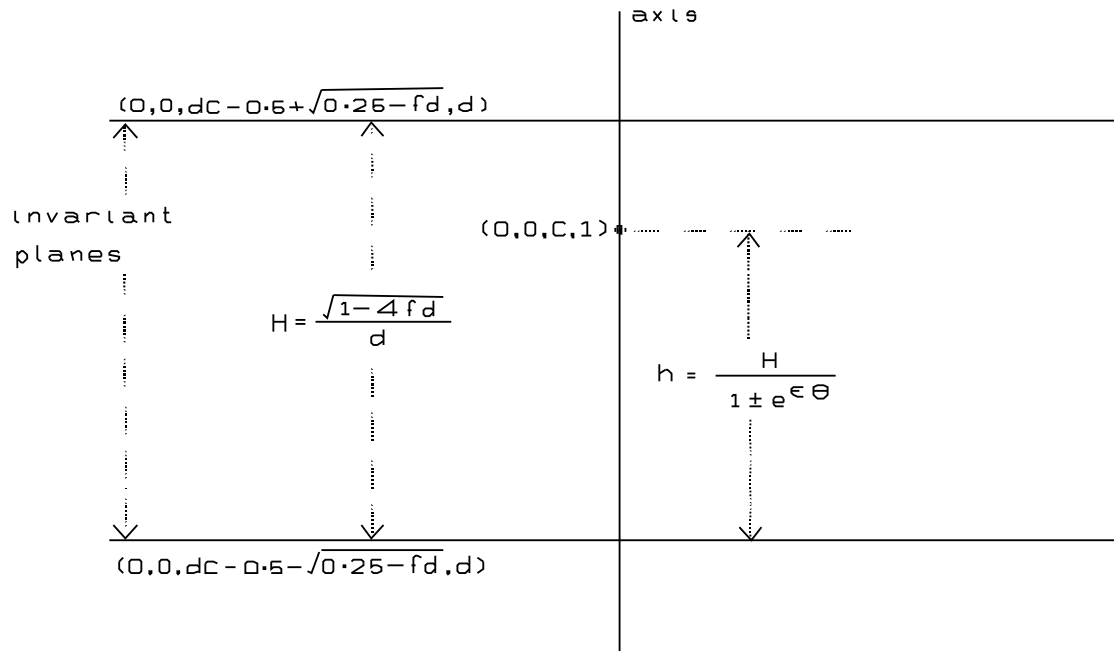
$$\varepsilon = \frac{\tanh^{-1}(1 - 4fd)^{\pm \frac{1}{2}}}{\tan^{-1} \left(\frac{f+d}{1-fd} \right)} \quad (10)$$

(the $\pm$ arising from the square in (9)).

A striking fact is the absence of c in the above equations. The parameters of the linear complex suffice to determine λ , ε and θ . Whatever the value of c the same path curve system is obtained, which is very satisfactory since the origin is an artifice of the coordinate system. It is important to consider c , however, as this enables us to locate the etheric centre. The equation for the height of the bud or vortex is, using (5) above and (10) of Annex 1:

$$H = \frac{\sqrt{1 - 4fd}}{d} \quad (11)$$

8. A diagrammatic summary is shown below:



$$\begin{aligned}
 \Lambda_1 &= (1-fd) - i.(f+d) \\
 \Lambda_2 &= (1-fd) + i.(f+d) \\
 \Lambda_3 &= 0.5-fd+\sqrt{0.25-fd} \\
 \Lambda_4 &= 0.5-fd-\sqrt{0.25-fd}
 \end{aligned} \tag{12}$$

$$\begin{aligned}
 Q_1 &= (-i,1,0,0) \\
 Q_2 &= (i,1,0,0) \\
 Q_3 &= (0,0,dc-0.5+\sqrt{0.25-fd},d) \\
 Q_4 &= (0,0,dc-0.5-\sqrt{0.25-fd},d)
 \end{aligned} \tag{13}$$

Now from (8) we see that

$$\begin{aligned}
 \varepsilon' &= \log \left(\pm \frac{1+\sqrt{1-4fd}}{1-\sqrt{1-4fd}} \right) \\
 &= 2 \tanh^{-1} \sqrt{1-4fd} \\
 &\quad \text{else} \\
 &= 2 \coth^{-1} \sqrt{1-4fd}
 \end{aligned} \tag{14}$$

For the first case we have

$$\begin{aligned}
\tanh^2\left(\frac{\varepsilon'}{2}\right) &= 1 - 4fd \\
\text{so} \\
4fd &= 1 - \tanh^2\left(\frac{\varepsilon'}{2}\right) = \operatorname{sech}^2\left(\frac{\varepsilon'}{2}\right) \\
i.e. \\
fd &= \frac{1}{4\cosh^2\left(\frac{\varepsilon'}{2}\right)}
\end{aligned} \tag{15}$$

Hence in this case fd is necessarily positive, and f and d are of the same sign. We shall see later that this applies only to vortices.

For the second possibility in (14) we get

$$fd = \frac{-1}{4\sinh^2\left(\frac{\varepsilon'}{2}\right)} \tag{16}$$

f and d are necessarily of opposite sign in this case (for real ε') which will be seen later to apply both to buds and vortices.

9. Referring to the diagram in paragraph 8, the relative height of the etheric centre is

$$\frac{h}{H} = \frac{dc - \frac{1}{2} - \sqrt{\frac{1}{4} - fd}}{\frac{d}{\sqrt{1-4fd}}} = \frac{1}{2} \left(1 + \frac{1}{\sqrt{1-4fd}} \right)$$

substituting for $\sqrt{1-4fd}$ from (14) we get

$$\frac{h}{H} = \frac{1}{2} \left(1 + \frac{1}{\tanh\left(\frac{\varepsilon'}{2}\right)} \right)$$

or

$$\frac{h}{H} = \frac{1}{2} \left(1 + \frac{1}{\coth\left(\frac{\varepsilon'}{2}\right)} \right)$$

Substituting for tanh and coth we get

$$\frac{h}{H} = \frac{1}{1 - e^{\varepsilon'}} = \frac{1}{1 - e^{\varepsilon\theta}} \quad \text{for } fd > 0$$

or

$$\frac{h}{H} = \frac{1}{1 + e^{-\varepsilon'}} = \frac{1}{1 + e^{-\varepsilon\theta}} \quad \text{for } fd < 0 \quad (17)$$

hence the relative height of the etheric centre is independent of λ .

10. λ changes sign, from equation (6), when

$$\sqrt{(1+f^2)(1+d^2)} = \left[\frac{1}{2} \pm \sqrt{\frac{1}{4} - fd} \right]^2 = \frac{1}{2} - fd \pm \sqrt{\frac{1}{4} - fd}$$

from which we get the quartic equation

$$\frac{f^2}{1+d^2} + 4f^3d + f^2 + 4fd + d^2 = 0 \quad (18)$$

This equation has two real roots in general which are listed in Table 3: two values of f for each d . It is clear that no solution to (18) exists if $fd > 0$, since all terms have an even number of occurrences of f or d . Hence λ cannot change sign, and since $\lambda < 0$ when $fd = 0.25$ we see that $\lambda < 0$ if $fd > 0$.

Equation (6) changes sign, for a given d , at the two values of f in table 3, so for $fd < 0$ we find $\lambda > 0$ between these values. In particular,

$\lambda > 0$ if $f = -d$ as no non-zero solution to (18) exists for $f = -d$, and $\lambda = +6.15$ for $f = -d = 1$.

This is summed up in fig. 3c.

11. It was shown how to determine λ , ε and θ from f and d in paragraph 7 above. To solve the converse problem we will use λ and ε' , the latter being $\theta\varepsilon$.

From (6) and (14) we have, for the case when $fd < 0$,

$$\lambda = - \frac{\frac{1}{2} \log[(1+f^2)(1+d^2)] - 2 \log \left[\frac{1 - \tanh\left(\frac{\varepsilon'}{2}\right)}{2} \right]}{\frac{1}{2} \log[(1+f^2)(1+d^2)] - 2 \log \left[\frac{1 + \tanh\left(\frac{\varepsilon'}{2}\right)}{2} \right]}$$

whence

$$(1+f^2)(1+d^2) = \left(\frac{1 - \tanh\left(\frac{\varepsilon'}{2}\right)}{2} \left(\frac{1 + \tanh\left(\frac{\varepsilon'}{2}\right)}{2} \right)^\lambda \right)^{\frac{4}{1+\lambda}} = \gamma, \text{ say.}$$

From (15):

$$f^2 + d^2 = \gamma - 1 - \frac{1}{16 \cosh^4\left(\frac{\varepsilon'}{2}\right)} = \varphi, \text{ say.} \quad (19)$$

Again from (15) we get

$$f^2 + \frac{1}{16 f^2 \cosh^4\left(\frac{\varepsilon'}{2}\right)} = \varphi$$

whence

$$f^2 = \frac{\varphi \pm \sqrt{\varphi^2 - \frac{1}{4 \cosh^4\left(\frac{\varepsilon'}{2}\right)}}}{2} \quad \text{for } fd > 0 \quad (20)$$

(19) is symmetrical for f^2 and d^2 , so any value of f satisfying (19) also counts as a value of d . Hence the $\pm$ distinguishes between f^2 and d^2 .

For $fd > 0$ we replace $\tanh(\varepsilon'/2)$ by $\coth(\varepsilon'/2)$, and fd by $-1/(\sinh^2 \varepsilon'/2)$, and the same process then gives

$$f^2 = \frac{\varphi \pm \sqrt{\varphi^2 - \frac{1}{4\sinh^4\left(\frac{\varepsilon'}{2}\right)}}}{2} \quad \text{for } fd < 0 \quad (21)$$

where

$$\varphi = \gamma - 1 - \frac{1}{16\sinh^4\left(\frac{\varepsilon'}{2}\right)}$$

and

$$\gamma = \left(\frac{1 - \coth\left(\frac{\varepsilon'}{2}\right)}{2} \left(\frac{1 + \coth\left(\frac{\varepsilon'}{2}\right)}{2} \right)^\lambda \right)^{\frac{4}{1+\lambda}}$$

Again d^2 is obtained by taking the opposite sign for the $\pm$ to that taken for f^2 .

Hence in principle we can find d and f , given λ and ε' . In practice some knowledge of the sign of fd is required to avoid ambiguity or imaginary solutions. However, ambiguity is of intrinsic interest as we will see later.

12. It follows from (20) and (21) that d and f need not be real, so the relationship between permitted values of λ , ε and θ needs investigation.

An alternative form of (20) and (21), useful for calculation, is easily shown to be

$$f^2 = \frac{\varphi \pm \sqrt{\varphi^2 - 4\gamma}}{2} \quad (22)$$

where

$$\varphi = \left[\left(e^{\varepsilon'} \pm 1 \right) \left(1 \pm e^{-\varepsilon'} \right)^\lambda \right]^{-\frac{4}{1+\lambda}} - 1 - \gamma$$

and

$$\gamma = \left[\left(e^{\varepsilon'} \pm 1 \right) \left(1 \pm e^{-\varepsilon'} \right) \right]^{-2} \quad (\text{redefining } \gamma)$$

taking $+$ when $fd > 0$, and $-$ when $fd < 0$.

The ambiguity of sign in ϕ and γ covers that between $\tanh$ and $\coth$ and between $\sinh$ and $\cosh$ previously. It is not easily resolved prior to calculation, so both must be tried in most cases. If λ is positive then fd is negative as we saw in the previous paragraph, but if λ is negative then sometimes two solutions are possible, and sometimes only one. This is because there are two kinds of vortex (see paragraph 16 below), one kind having fd positive and the other negative.

For real f and d we require in (22) that

$$\phi^2 - 4\gamma > 0 \quad \text{and} \quad \phi > 0 \quad (\text{since } \gamma \text{ necessarily } > 0)$$

$$\text{so } (\phi - 2\sqrt{\gamma})(\phi + 2\sqrt{\gamma}) > 0$$

If $\phi > 2\sqrt{\gamma}$ then we secure both positive ϕ and a real discriminant in (22)

$$\text{so } [(e^{\epsilon'} \pm 1)(1 \pm e^{-\epsilon'})^\lambda]^{-4/(1+\lambda)} > 1 + \gamma + 2\sqrt{\gamma} = (1+\sqrt{\gamma})^2$$

$$\text{Let } y = P^{-4/(1+\lambda)} Q^{-4\lambda/(1+\lambda)} - [1 + 1/(PQ)]^2$$

where $P = (e^{\epsilon'} \pm 1)$ and $Q = (1 \pm e^{-\epsilon'})$ and so $\sqrt{\gamma} = 1/(PQ)$

$$\text{then } \frac{dy}{d\lambda} = \frac{4}{(1+\lambda)^2} P^{-4/(1+\lambda)} Q^{-4\lambda/(1+\lambda)} \log P + \frac{4}{(1+\lambda)^2} P^{-4/(1+\lambda)} Q^{-4\lambda/(1+\lambda)} \log Q$$

$$\text{so } \frac{dy}{d\lambda} = \frac{4}{(1+\lambda)^2} P^{-4/(1+\lambda)} Q^{-4\lambda/(1+\lambda)} (\log P + \log Q)$$

Now ϵ' is necessarily positive (as ϵ and θ have the same sign; cf also (8)), so P and Q are both positive.

If λ_c is the value of λ that makes $y = 0$, then it follows that for real d^2 and f^2 we require $\lambda > \lambda_c$, since $dy/d\lambda$ is positive. This applies regardless of the sign ambiguity.

λ_c is found by setting $y = 0$,

$$\text{so } P^{-4/(1+\lambda)} Q^{-4\lambda/(1+\lambda)} = [1 + 1/(PQ)]^2$$

and taking logs gives

$$-[4/(1+\lambda)] \log P - [4\lambda/(1+\lambda)] \log Q = 2 \log [1 + 1/(PQ)]$$

$$\text{whence } \lambda_c = \frac{2 \log P + \log(1+1/PQ)}{2 \log Q + \log(1+1/PQ)} \quad (23)$$

where $P = (e^{\epsilon'} \pm 1)$ and $Q = (1 \pm e^{-\epsilon'})$

We note that λ_c makes the discriminant in (22) zero, so $d^2 = f^2$ in the critical case. Values of λ_c and ϵ_c' are tabulated in table 1 and plotted in fig 2.

If $fd < 0$ then $d = -f$ so $\tan \theta_c = 0$ (cf (7)). Hence $\theta = n\pi$.

Referring to Annex 1 para 4 we see that if $y = 0$ then $x' = \alpha x$ and $y' = -\beta x$

Now $\alpha = 1 - fd > 0$ since $0.25 > fd$ for real λ and ϵ . Therefore the quadrant of θ depends only upon $-\beta = -(f + d)$. It follows that θ is necessarily in the first or fourth quadrant so odd values of n do not arise, and for even values we merely repeat with added complete rotations, so consideration of $n = 0$ suffices.

Thus $\tan \theta = 0$ implies $\theta = 0$, so

$$\epsilon_c = \epsilon_c' / \theta_c = \epsilon_c' / 0 \text{ which tends to infinity}$$

Since for a given value of ϵ we require $\lambda > \lambda_c$, conversely for a given value of λ we require $\epsilon < \epsilon_c$ (if $\lambda > 0$). Hence for buds all possible combinations of λ and ϵ may occur, but θ is constrained by that choice (recall that $\lambda > 0$ when $f = -d$).

If $fd > 0$, $d = f$ so $\tan \theta \leq 0$ in general.

This time, for a given λ , $\epsilon_c = \epsilon'_c/\theta_c < \text{infinity}$, i.e. not all combinations of λ and ϵ are possible.

Table 2 shows the relation between critical values of λ and ϵ , compared with the relation between λ and ϵ required for path curves to be the asymptotic curves of a vortex (recall that $fd > 0$ implies $\lambda < 0$). The results are plotted in Fig. 1. The asymptotic cases thus cannot arise in the present context. It is clear from Fig. 1 that we require

$$|\epsilon| > |\epsilon_c| \text{ for a given } \lambda.$$

13. It has been shown how f and d may be derived from λ and ϵ' . Since we have two parameters for the complex and three for the path curve system, the latter are interdependent. We may choose any legal pair of λ , ϵ , θ freely, and the third is then determined along with f and d . The "legality" is purely determined by the requirement for a semi-imaginary invariant tetrahedron. "Illegal" combinations yield fully imaginary tetrahedra, which are of course perfectly valid mathematically.

14. If we know ϵ and θ , then we have

$$fd = \sqrt{\gamma} \text{ (from equations (22))}$$

and

$$\tan \theta = -\frac{f+d}{1-fd} \quad \text{from (7)}$$

$$= -\frac{d+\sqrt{\gamma}/d}{1-\sqrt{\gamma}} = t, \text{ say}$$

$$\text{so } d^2 + t(1-\sqrt{\gamma})d + \sqrt{\gamma} = 0, \text{ and}$$

$$d = \frac{(\sqrt{\gamma}-1)\tan \theta \pm \sqrt{(\sqrt{\gamma}-1)^2 \tan^2 \gamma}}{2} \quad (24)$$

f again takes alternate values, and $\gamma = [(e^{\theta\epsilon} \pm 1)(1 \pm e^{-\theta\epsilon})]^{-2}$
Hence f and d are determined, and λ is found from (6).

15. To find f and d from λ and θ or from λ and ϵ requires the solution of transcendental equations. It is best accomplished graphically or iteratively on a computer, and Fig. 3 shows λ and ϵ loci on an f/d graph to give a feel for the constraints.

16. The position of the etheric centre was briefly discussed in para. 9. It is clear that when $fd < 0$ it lies between the real invariant planes, and in particular lies inside bud forms since h/H is then positive (cf (17)).

If $d > 0$ then from (11) we see that $H > 0$, so Q_3 lies above Q_4 (cf diagram in para. 8). The conventions adopted require $|\lambda| > 1$, i.e. treating Q_3 as the uppermost invariant point, then the situation is as shown in Fig. 4 for buds and vortices. Since $1 > h/H > 0.5$ in this case, the etheric centre is nearer to the "sharper" end of the egg, or to the apex of the vortex.

We note that there are two kinds of vortex, one with its etheric centre outside the invariant planes and the other with it between them (Fig. 4). The latter has a cone of contact in the etheric centre i.e. it possesses tangent planes "at infinity" as viewed from counterspace. If $fd > 0$ the Chief Parameter of the complex is negative (Ref 3 page 31) which means that for increasing θ we are moving down the spirals of the complex (Ref 3 page 32) i.e. the movements along the path curve and complex spirals are opposite. If, on the other hand, $fd < 0$ then those movements are in the same vertical sense, which applies to eggs and some vortices.

SPECIAL CASES

17. If $f = -d$ we have $\theta = 0$ (para. 11), and $\Lambda_1 = \Lambda_2$ from (12). Here we lose the semi-imaginary invariant tetrahedron, and obtain a line of real self-corresponding points at infinity (cf ref. 5 page 349). Hence each plane in the vertical axis is invariant, and the path curves are profiles in those planes, with $\varepsilon = \infty$ and $\theta = 0$ (c.f.(7) and (10)).

If $fd = 0.25$, $\Lambda_3 = \Lambda_4$, the vertical axis becomes a line of self-corresponding points, and since the two distinct ones remain the circling points at infinity, we have a pure rotation (c.f. Ref. 5 page 350).

MORE GENERAL CASES

18. In deriving the transformation (4) we assumed particularly that the same linear complex was to be used in constructing both correlations, first from its pointwise and then its planewise aspect. We could generalise in several steps:

- a. To use distinct coaxial complexes, one for each correlation;
- b. To use distinct complexes with intersecting axes;
- c. To use distinct complexes with skew axes.

Most effort has been expended on the identical-complex case as it is "strongest" i.e. involves fewest parameters.

19. If we replace d and f by a and b in (3) we obtain a distinct complex $B_{i,k}'$. This gives a transformation

$$\begin{bmatrix} 1-bd & b+d & 0 & 0 \\ -(b+d) & 1-bd & 0 & 0 \\ 0 & 0 & 1-a(f+c) & f-c+ac^2 \\ 0 & 0 & -a & -a(f-c) \end{bmatrix} \quad (25)$$

Then

$$\begin{aligned} \Lambda_1 &= (1-bd) - i.(b+d) \\ \Lambda_2 &= (1-bd) + i.(b+d) \\ \Lambda_3 &= 0.5-af + \sqrt{\{0.25-af\}} \\ \Lambda_4 &= 0.5-af - \sqrt{\{0.25-af\}} \end{aligned} \quad (26)$$

and the invariant points are

$$\begin{aligned} Q_1 &= (-i, 1, 0, 0) \\ Q_2 &= (i, 1, 0, 0) \\ Q_3 &= (0, 0, ac-0.5 + \sqrt{\{0.25-ad\}}, a) \\ Q_4 &= (0, 0, ac-0.5 - \sqrt{\{0.25-ad\}}, a) \end{aligned} \quad (27)$$

In general we obtain the same results: a semi-imaginary invariant tetrahedron, with similar expressions for λ , ε and θ , and an identical expression for h/H :

$$\lambda = -\frac{\log \sqrt{(1+b^2)(1+d^2)} - 2\log \left[\frac{1}{2} - \sqrt{\frac{1}{4} - af} \right]}{\log \sqrt{(1+b^2)(1+d^2)} - 2\log \left[\frac{1}{2} + \sqrt{\frac{1}{4} - af} \right]}$$

$$\varepsilon' = \frac{1}{2} \log \left(\frac{1 + \sqrt{1-4af}}{1 - \sqrt{1-4af}} \right)^2$$

$$\theta = -\tan^{-1} \left(\frac{b+d}{1-bd} \right)$$

$$\frac{h}{H} = \frac{1}{1 \pm e^{\mp \varepsilon \theta}}$$
(28)

Thus the relative height of the etheric centre is determined solely by θ and ε as before, and c is not involved in these expressions. In all essentials the results are the same. However, there are now no restrictions on λ , ε and θ so asymptotic path curves on vortices are possible.

Also, biaxial collineations are additionally possible as a special case (when $b = -d$ and $af = 0.25$) i.e. we obtain radial expansion about the axis (cf Ref. 5 page 351). The path curves are the lines of a linear congruence. Space homologies are also possible by setting

$$b = -d, \text{ and } b^2 = (0.5 \pm \sqrt{0.25 - af})^2 - 1$$

yielding three identical eigenvalues. This gives radial expansion about the isolated self-corresponding point, the path curves being the lines of a star.

Neither the biaxial collineation nor the homology may be involutory for real values of the parameters, and space elations are not obtainable.

20. The other general cases will not be considered here as they have more complicated matrices that fall outside the scope of the general theory in Annex 1. Their characteristic equations are not necessarily factorisable algebraically which defeats a general treatment such as above.

CONTACT CORRELATIONS

21. Suppose $\mathbf{K}$ is a contact correlation.
Then the plane $\boldsymbol{\pi}$ obtained from the point $\mathbf{x}$ is

$$\boldsymbol{\pi} = \mathbf{K} \mathbf{x} \quad (\boldsymbol{\pi} \text{ and } \mathbf{x} \text{ are vectors}).$$

$$\text{But, } \boldsymbol{\pi}^T \mathbf{x} = 0 \quad \text{for a contact correlation.}$$

$$\text{so } \mathbf{x}^T \mathbf{K}^T \mathbf{x} = 0 \quad \text{for all } \mathbf{x}.$$

giving $k_{ij} = 0$ for $i = j$ (k_{ij} are the elements of $\mathbf{K}$)
and $k_{ij} = -k_{ji}$ otherwise

i.e. we have a null polarity.

Thus null polarities are the only linear contact correlations. This shows that the transformations considered so far do not involve contact correlations, so the "etheric forms" or Counterspace forms enveloped by planes corresponding to the pointwise buds and vortices do not coincide with the latter. Such a possibility does not appear to arise from the method of inter-relating space and Counterspace used here. Even a coincidence of the overall surfaces is not obtainable (this is seen by considering the eigenplanes of the inverse collineation from counterspace to counterspace).

Etheric Centre of Cosmic Vortex

In articles cs1.odt, cs2.odt and cs3.odt the etheric centre of a cosmic vortex was not worked out. A general path curve transformation is given by

$$\begin{bmatrix} r \cos \theta & r \sin \theta \\ -r \cos \theta & r \sin \theta \\ a & b \\ d & e \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} r(x \cos \theta + y \sin \theta) \\ r(y \cos \theta - x \sin \theta) \\ az + b \\ dz + e \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}$$

so $z' = \frac{az+b}{dz+e}$ is a general projectivity between the double points.

When $z=0$, $z'=b/e=0$ so $b=0$;

when $z=H$, $z'=H = \frac{aH+b}{dH+e}$

so $dH^2 + (e-a)H - b$

so for $b=0$, $dH^2 + (e-a)H = 0$

i.e. $dH + e - a = 0$

Let the radial expansion factor be

$$\rho = \frac{r}{dz+e} = \frac{r}{dH+e} \text{ for } z=H, \text{ so } .$$

$$r = d\rho H + \rho e = a\rho, \text{ so } a = r/\rho$$

giving d in terms of e :

$$dH + e - a = dH + e - \frac{r}{\rho} = 0$$

$$\text{whence } d = \frac{r - \rho e}{\rho H}$$

giving the transformation matrix as

$$\begin{bmatrix} r \cos \theta & r \sin \theta \\ -r \sin \theta & r \cos \theta \\ r/\rho & 0 \\ \frac{r - \rho e}{\rho H} & e \end{bmatrix}$$

Thus

$$\begin{bmatrix} r \cos \theta & r \sin \theta & r/\rho & 0 \\ -r \sin \theta & r \cos \theta & \frac{r-\rho e}{\rho H} & e \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} r(x \cos \theta + y \sin \theta) \\ r(y \cos \theta - x \sin \theta) \\ \frac{r z}{\rho} \\ \frac{r-\rho e}{\rho H} z + e w \end{bmatrix}$$

$H=\infty$ for a cosmic vortex then gives

$$\begin{bmatrix} r(x \cos \theta + y \sin \theta) \\ r(y \cos \theta - x \sin \theta) \\ \frac{r z}{\rho} \\ e w \end{bmatrix} = \begin{bmatrix} \frac{r}{e w} (x \cos \theta + y \sin \theta) \\ \frac{r}{e w} (y \cos \theta - x \sin \theta) \\ \frac{r z}{\rho e w} \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{r}{e} (x \cos \theta + y \sin \theta) \\ \frac{r}{e} (y \cos \theta - x \sin \theta) \\ \frac{r z}{\rho e} \\ 1 \end{bmatrix} \text{ for } w=1$$

Thus $z' = \frac{r z}{\rho e}$ giving a multiplier $m = \frac{r}{\rho e}$ for the vertical movement per step. This pure multiplication characterises a cosmic vortex and if $m > 1$ then $z' \rightarrow \infty$ i.e. to the upper invariant plane at infinity, while if $0 < m < 1$ then $z' \rightarrow 0$ i.e. to the lower invariant plane.

Thus with $H = \infty$ we may equate this transformation to that in cs1.odt:

$$\begin{bmatrix} r \cos \theta & r \sin \theta & r/\rho & 0 \\ -r \sin \theta & r \cos \theta & 0 & e \end{bmatrix} = \begin{bmatrix} 1-fd & f+d & & \\ -(f+d) & 1-fd & & \\ & & 1-d(f+c) & fc-dc^2 \\ & & -d & -d(f+c) \end{bmatrix}$$

noting that d has a different meaning from earlier. Thus $fc - dc^2 = 0$ giving $c = f/d = \infty$ for $d=0$ i.e. the etheric centre at $(0,0,c,1)$ is in the invariant plane at infinity.