Lemniscate Path of Sun and Earth

The following pages contain a scanned copy of an article written in 1975 presenting a purely projective approach to Rudolf Steiner's indications that the Sun and Earth move on a lemniscate. It assumes the reader is familiar with another article by the author describing so-called "I-Space", where a realisation of projective geometry is used based on circles and points instead of lines and points. For convenience this is summarised in the Appendix at the end, and may be read first.

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AN APPROACH TO THE LEMNISCATE-PATH OF THE SUN AND EARTH

by
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In an article in issue number 10 of this correspondence (ref 1) the writer set forth the foundations of point/conic projective geometry,† and the special case called "I-space" in which the lemniscate may act as a "conic." It is the writer's belief to date that I-space is the only one in which this is possible, as the two absolute circling points at infinity (see ref 7) must be base points of the p/c space since the lemniscate has a self-crossing point in each (any contrary thoughts would be welcomed!). Hence the stress laid on I-space at the end. The wish to treat the lemniscate as a "conic" springs from the desire to approach Rudolf Steiner's astronomical indications with the aid of projective concepts to assist in the analysis. The aim at present is to develop a "vocabulary" or "language" with which to discuss the problem. The present article presents one solution of the Sun/Earth lemniscate path, suggested by Steiner, not as a rigorous theory, but as a proposal to stimulate thought and indicate the direction being pursued: to see how the "vocabulary" may be used to discuss the problem.

At the outset it seemed evident that a careful distinction must be made between "Maya" and "Truth." Whatever lemniscate approach we adopt we must be able to "save the appearances" (as Owen Barfield would say), and those appearances are the point of view of Maya. As a tentative beginning we will attempt to save two essential appearances:

1) that the fixed stars appear indeed to be fixed
2) that the Sun appears to move continuously round the Zodiac.

2) that the Sun appears to move continuously round the Zodiac. Later we must ensure other aspects such as the more or less constant apparent size of the Sun throughout the year, and so forth.

We shall work with I-space for the reasons already stated. Let us first check appearance number one, that the fixed stars are fixed. We take light rays to follow "straight lines," which are circles through the point $\bar{Z}$ (which we called the real base point (RBP) of the I-space concerned). Let there be two fixed stars $\bar{S}_1, \bar{S}_2$ which will be seen from $\bar{E}$ (the Earth) by means of light rays α, β which are the unique circles through $\bar{S}_1, \bar{E}, \bar{Z}$ and $\bar{S}_2, \bar{E}, \bar{Z}$ respectively. Now a word about Maya: we take the stars $\bar{S}_1, \bar{S}_2$ to appear from the Earth to be in the directions of the two tangents to α and β at $\bar{E}$. They will be at some angle θ. By elementary geometry the tangents $\bar{a}, \bar{b}$ to α, β respectively at $\bar{Z}$ are also inclined at the same angle θ. Now, whatever path $\bar{E}$ may follow the angle θ must remain constant (refinements to allow for the proper motion of the stars can come later). This is true if we adopt the simplest metrical definition of angle, which we will

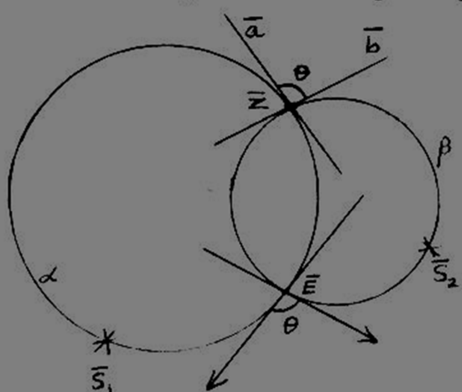


FIGURE ONE

† [For those without a copy of no. 10 at hand, the definitions again: Since in ordinary projective geometry any five points determine a conic, let three (non-collinear) base points be given; then any two further ordinary points determine a conic which now may play the role of a "line," and any two such "lines" intersect in a single further point. In "I-space" two of the base points are chosen to be the complex conj. pts. at ∞. — Ed.]

for the time being. Now $(\bar{a}),(\bar{b})$ are equal pencils as θ is constant. The situation in the ordinary plane imaged by figure 1 is shown in figure 2. $\bar{a}$ always intersects α in Z, so a,a' (the lines imaged by $\bar{a}$,α respectively) meet on the line at infinity (joining the two imaginary poles; see ref 1) and so are parallel. Similarly b,b' are parallel. Thus since (a),(b) are directly equal pencils it follows that (a'),(b') are too, and hence the locus of E is a circle. Thus the locus of $\bar{E}$ is also a circle (property 3 of I-space; see ref 1 page 21). The lemniscate is out of the question as an orbit of the Earth! Therefore we reject the concept that the fixed stars have fixed locations and adopt instead Steiner's suggestion that they be regarded as indicating directions only (ref 2). How should we do this? One way of defining a direction is by a set of parallel lines, which are imaged in I-space by the set of circles tangential to a fixed line $\bar{s}$ at $\bar{Z}$, where $\bar{S}$ is the image of the line s through Z parallel to the original set of

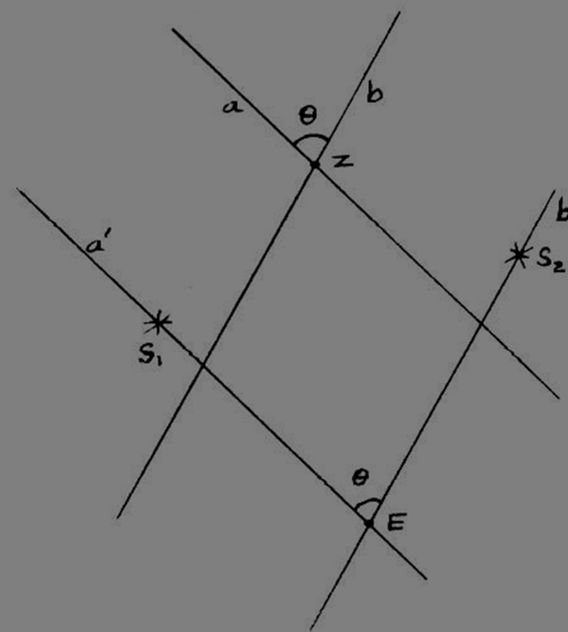


FIGURE TWO

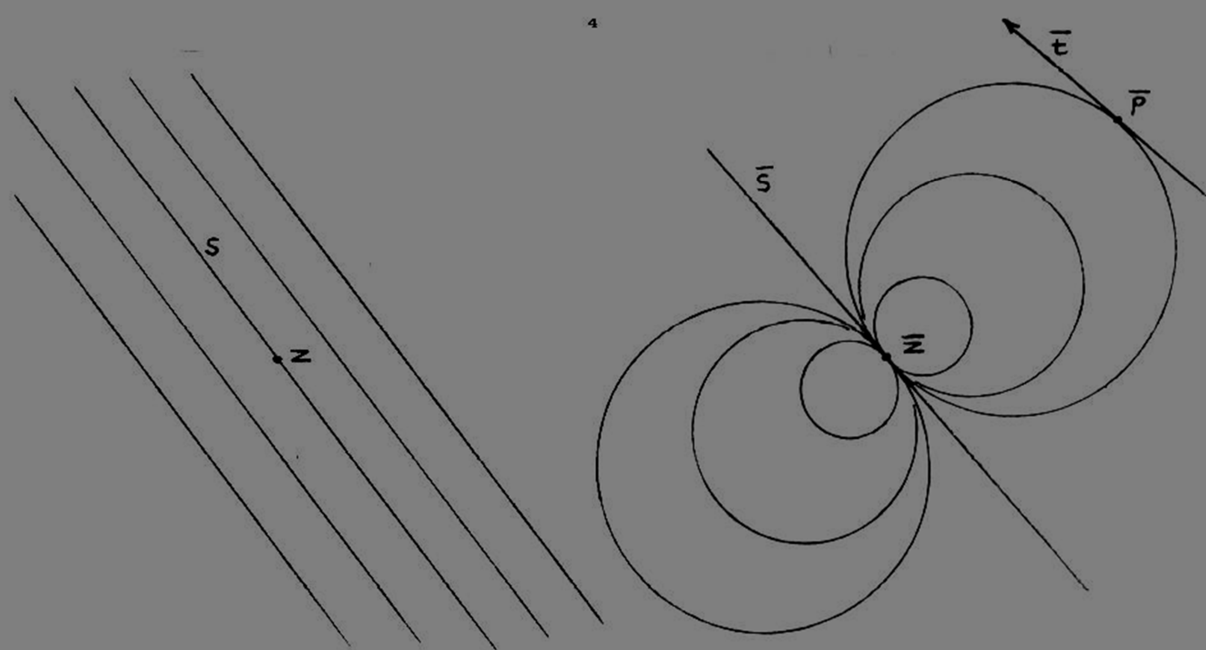


FIGURE THREE

lines (fig 3). We say that a fixed star now has no location, but is represented by such a set of circles in I-space (including $\bar{s}$). At a point $\bar{P}$ in I-space the star is seen in the direction of the tangent $\bar{t}$ to the unique circle of the set containing $\bar{P}$.

Consider now two stars represented by $\overline{S}_1$ and $\overline{S}_2$ (and their attendant circles), and let the Earth be at $\overline{E}$ (fig 4). $\overline{S}_1, \overline{S}_2$ will be inclined at a fixed angle θ, and the tangents to the two circles containing $\overline{E}$, each belonging to one star, will now be at the same angle θ no matter where $\overline{E}$ is. Hence whatever path $\overline{E}$ follows two fixed stars will always appear to be in directions separated by a constant angle. This takes care of appearance number one from a two dimensional point of view.

Now we come to appearance number two! First we will adopt the lemniscate, without apology, as the mutual Sun/Earth orbit in I-space so that it is a "conic." There are, of course, numerous ways of letting the Sun and Earth move round it, and a reasonable starting point is to consider them to follow two ranges "in involution" on the "conic." Later we may adopt two "projective ranges" instead if desirable. We shrink from adopting two "non-projective ranges" at present!

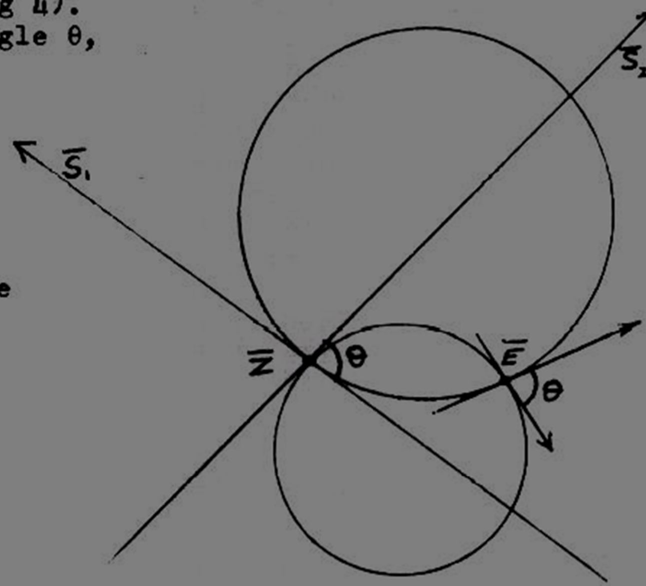


FIGURE FOUR

Pursuing the "involution" approach, we stipulate further that it be a "circling involution" ("elliptic involution") to ensure that the double-points are not real (we do not particularly wish to collide with the Sun, at least not yet!). A circling involution on a conic is generated by the rays of a flat pencil whose centre lies "inside" the lemniscate (fig 5). Each circle intersects the lemniscate in two points, one of which is the position of the Sun and the other that of the Earth. $\overline{C}$ is the centre of the "pencil." Consider now one circle α of the "pencil," whose tangent at $\overline{Z}$ is $\overline{t}$ (see fig 6, facing page). We choose to let $\overline{t}$ rotate uniformly about $\overline{Z}$, constraining α to vary accordingly. $\overline{S}$ and $\overline{E}$ will now move round the lemniscate smoothly and continuously. Let there be some fixed star represented by $\overline{s}$. Then the angle θ between $\overline{s}$ and $\overline{t}$ will increase uniformly as $\overline{t}$ rotates. If β is the circle of the fixed star containing $\overline{E}$, then it follows that the two tangents to α and β at $\overline{E}$ are also separated by an angle that increases uniformly. Hence the Sun appears to move steadily round the Zodiac, which saves appearance number two.

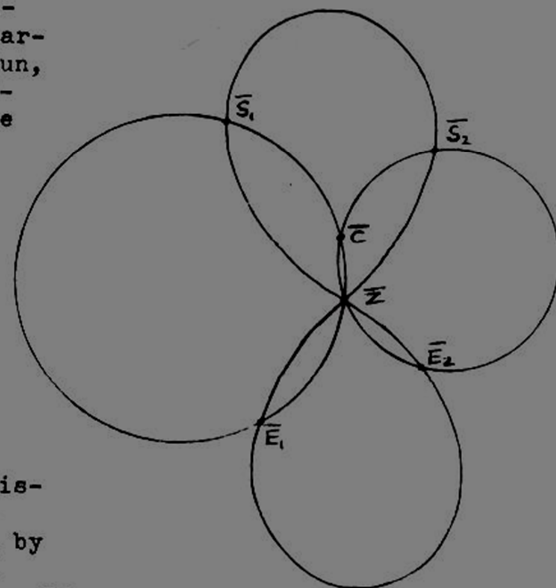


FIGURE FIVE

We must at this point decide in which of the two possible directions a star is seen as determined by a tangent such as $\overline{t}$ in fig 3, and also in which direction the Sun is seen from the Earth. The conventions we adopt are:

1) The lines such as $\bar{s}$ in figure 3 are given a definite sense, and then the sense round a circle of the star's system is continuous and consistent with that of $\bar{s}$ at $\bar{Z}$.

2) The Sun is always seen from the Earth in the outward direction from the lemniscate along a tangent such as that at $\bar{E}$ to α in figure 6.

These two conventions ensure that the angle relationships vary consistently round the lemniscate.

This picture is, of course, too idealised. In reality the stars possess proper motions which may be allowed for by letting their representative lines such as $\bar{s}$ rotate slowly. Also, the Sun varies in speed round the Zodiac in the course of the year (due to the eccentricity of the Earth's orbit, in Copernican terms). To allow for this we may let $\bar{t}$ rotate in an appropriate manner, or we may perhaps choose a space that is very nearly I-space, but not quite. The metric adopted to interpret θ is open to variation, too.

Now we seem to have too much freedom! Such abundance of freedom is normally considered poor by scientific standards. Constraints are desired which impose observational predictions capable of being checked. There are two points to be made here: Firstly that this view is not intended to be scientific in the strict sense of the word yet, it is only an approach to the problem, a discussion of it really; and secondly that the above freedom will anyway rapidly erode once we attempt to construct the complete Solar System!

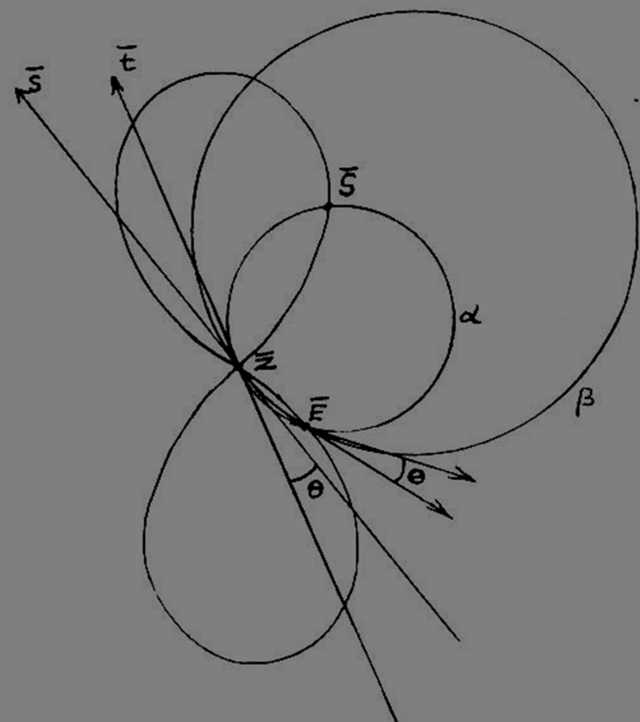


FIGURE SIX

Several suggestions and hints made by Steiner are brought together in the above approach. First, of course, that the Sun and Earth travel on a mutual lemniscate orbit (refs 3 - 5 and many others). Secondly that space should be considered as "curved" rather than Euclidean (we do not mean to imply that this was an original suggestion of Steiner's) (ref 3), which the introduction of I-space implies; we seem to have two spaces, one similar to George Adams' negative space, and then what we may call "Maya space" which is quasi-Euclidean in terms of angle definition and straight-line viewing. Thirdly, that light rays are "elastic," returning to the Sun after a while (ref 6) which is enshrined in the "straight lines" of I-space being circles. Fourthly that the fixed stars do not have a location in space, but should be regarded as indicating directions (ref 2).

What about Steiner's "screw-like spiral"? This we may obtain by supposing that the plane of the orbit moves parallel to itself. The points occupied by the Sun and Earth respectively will then be in different planes so that they may follow the same spiral line (ref 3), occupying conjugate positions of the "involution" only when seen in perspective. That the "involution" is circling is essential in this respect, or a kind of double-spiral would be obtained rather than a single one. This is still under investigation; indeed ref 5 indicates that we may have to choose two inclined planes instead.

A suggestion of Steiner's is not met by the proposed system: that whenever the Sun is at $\overline{Z}$ the Earth is at $\overline{D}$ or $\overline{D}'$ (fig 7) and vice versa (ref 5). The writer has investigated this, and it turns out that there is a unique pair of "projective ranges" with imaginary double-points that meet this added condition. The derivation is too lengthy to be included here (for lack of space) as it is necessary to construct a figure-of-eight "conic" having imaginary double contact with the lemniscate on the axis $\overline{b}$. It turns out that the required figure (Ω) is as shown in figure 8, passing through the foci of the lemniscate.

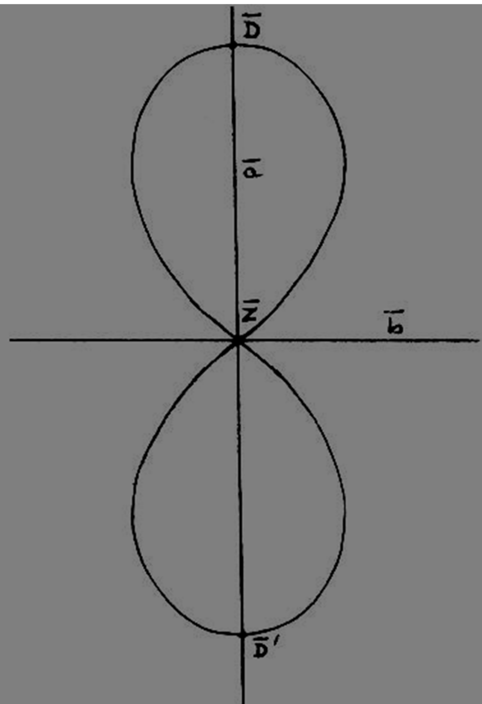


FIGURE SEVEN

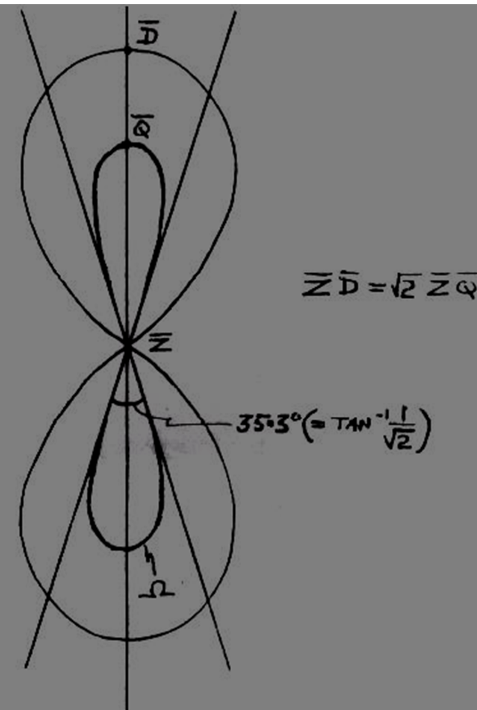


FIGURE EIGHT

The Sun/Earth positions are found from circles through $\overline{Z}$ tangential to Ω, each meeting the lemniscate in two points. The degrees of freedom are drastically reduced! The snag is that appearance number two is not saved, as the Sun periodically reverses its apparent direction of motion through the Zodiac, which is rather inconvenient! Therefore to incorporate Steiner's extra hint we need to re-define our metric somehow, if possible.

An attempt has been made to interpret Steiner's astronomical indications, an undertaking not without importance according to ref 8. Hopefully it has been demonstrated that the development of an appropriate "language" such as the theory of p/c space opens up possibilities for doing so, besides raising many further questions on such topics as the relationship between "Maya" and "reality."

Post-Script: In the above article several conventions have been adopted which may appear to be arbitrary and in violation of Occam's Razor. At some stage later in the work their significance and justification must be sought more deeply, of course. As for Occam's Razor, we must retain the Copernican theory if we insist on it and abandon any attempt to seek an alternative along the lines suggested by Steiner!

References

1. "Representations of the Projective Plane," Nick Thomas, Math.-Phys. Corresp. No. 10 (Christmas, 1974).

2. Rudolf Steiner, lecture at the Hague, 17th November 1923 (afternoon), contained in publication "Supersensible Man."

3. Rudolf Steiner, lecture course, "The Relation of the Diverse Branches of Natural Science to Astronomy," Stuttgart, 1 - 18 January 1921.

4. Rudolf Steiner, lecture course, "Man: Hieroglyph of the Universe," 9 April to 16 May 1920, Dornach.

5. Rudolf Steiner, lecture entitled "Das Karma des Geistschlafes und Traumlebens der verflossenen Jahrhunderte und unsere leidvolle Gegenwart," part of "Kosmische und Menschliche Geschichte," III, Dornach. Extract reprinted in Mathematisch-Astronomische Blätter, Heft 4, November 1942.

6. Rudolf Steiner, lecture, 31 August 1923, Penmaenmawr, printed as part of course "The Evolution of Consciousness," lecture XIII.

7. "Three Archetypal Scales," Stephen Eberhart, Math.-Phys. Corresp. No. 2 + 3 (Winter and Spring 1972-73).

8. Rudolf Steiner, lectures, published under title "Lucifer and Ahriman," given in Dornach and Berne, November 1919.

Appendix
A Brief Summary of I-Space

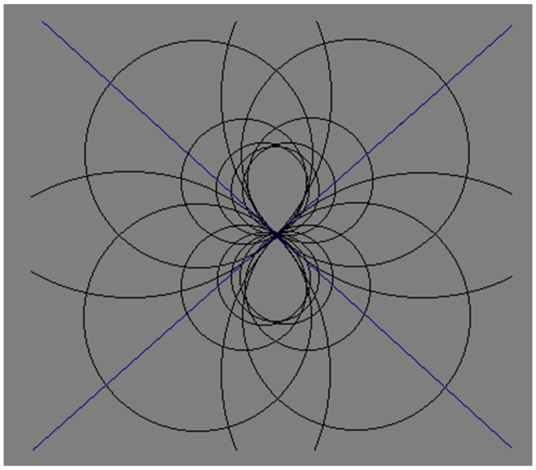
Projective geometry is based on axioms relating to fundamental undefined elements. These are termed "lines" (straight lines) and "points", and are usually identified with the intuitive versions of those concepts. For two-dimensional projective geometry we require that

1. Two distinct lines meet in exactly one point
2. Two distinct points are the join of exactly one line

where ideal points at infinity are included to cover parallel lines (in fact parallel lines do not exist in the pure projective plane).

The projective plane is derived from the Euclidean one by augmenting the latter with an ideal line *j* such that parallel lines meet on *j*. A metric is introduced by defining two imaginary absolute points I and J on *j* such that any two lines meeting *j* in points A and B respectively where A and B separate I and J harmonically are said to be orthogonal. Then all ellipses containing I and J are called circles. The lemniscate of Bernoulli contains both those points as imaginary crossing points.

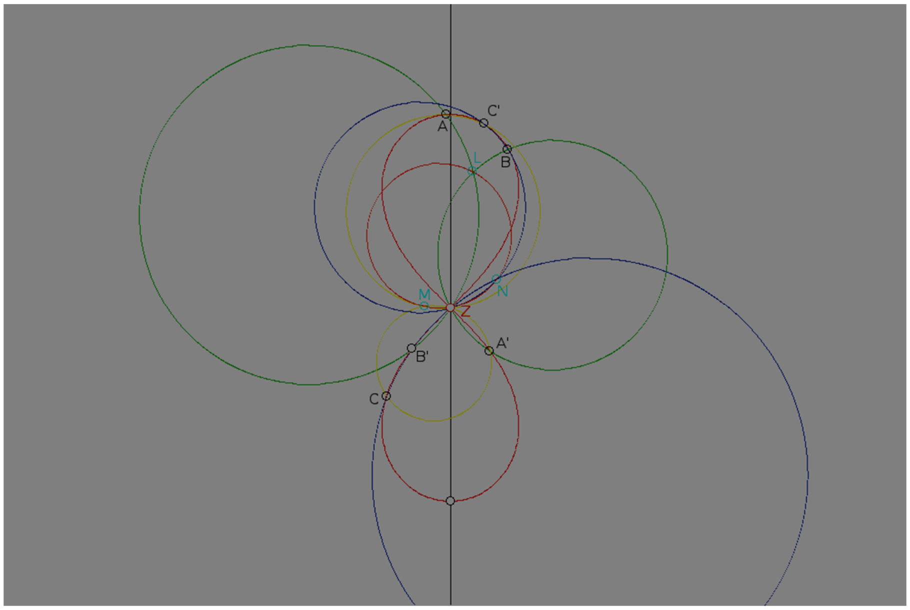
We may replace the intuitive elements called "lines" by circles through a fixed point Z. Then any two points determine exactly one circle in Z ("line"), and any two circles in Z ("lines") meet in exactly one point apart from Z. The following projective construction of a "conic" in this scheme is seen to generate a lemniscate, which is accordingly an example of such a "conic", and generally "conics" are fourth-order curves except in special cases.



Here two projective ranges on the two blue lines, when joined by circles in Z, yield the figure shown. The construction is well known (e.g. E.H. Lockwood *A Book of Curves*) - although not viewed as above - and generates the lemniscate of Bernoulli.

The "I-Space" of the article is such a geometry based on points and circles in a fixed point. It was chosen because it contains the lemniscate as a "conic".

As an example of how this geometry works, the following figure shows Pascal's Theorem in I-Space:



Six points A B C A' B' C' are shown on the lemniscate ("conic"). The two light green circles joining AB' and A'B intersect in L, the two light-blue circles BC' and B'C intersect in N and the two yellow circles joining AC' and A'C meet in M. All joins assume the circles contain Z of course. The three points LMN lie on an orange circle containing Z which is thus a "line" as required by Pascal's Theorem.