

AN APPROACH TO THE LEMNISCATE-PATH OF THE SUN AND EARTH

by
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In this article an attempt is presented (first published in Ref 1) to see how the Sun and Earth may follow a lemniscatory path as indicated by Rudolf Steiner (Refs 3 and 4). It is based on projective geometry rather than conventional dynamics, and uses a version of projective geometry in which the lemniscate of Bernoulli plays the role of a "conic section". That version of projective geometry will briefly be presented first.

Projective I-Space

Projective geometry is based on axioms relating to fundamental undefined elements. These are termed "lines" (straight lines) and "points", and are usually identified with the intuitive versions of those concepts. For two-dimensional projective geometry we require that

1. Two distinct lines meet in exactly one point
2. Two distinct points are the join of exactly one line

where ideal points at infinity are included to cover parallel lines (in fact parallel lines do not exist in the pure projective plane).

The projective plane is derived from the Euclidean one by augmenting the latter with an ideal line j such that parallel lines meet on j . A metric is introduced by defining two imaginary absolute points I and J on j such that any two lines meeting j in points A and B respectively where A and B separate I and J harmonically are said to be orthogonal. Then all ellipses containing I and J are called circles. The lemniscate of Bernoulli contains both those points as imaginary crossing points.

We may replace the intuitive elements called "lines" by circles through a fixed point Z . Then any two points determine exactly one circle in Z ("line"), and any two circles in Z ("lines") meet in exactly one point apart from Z . The following projective construction of a "conic" in this scheme is seen to generate a lemniscate, which is accordingly an example of such a "conic", and generally "conics" are fourth-order curves except in special cases.

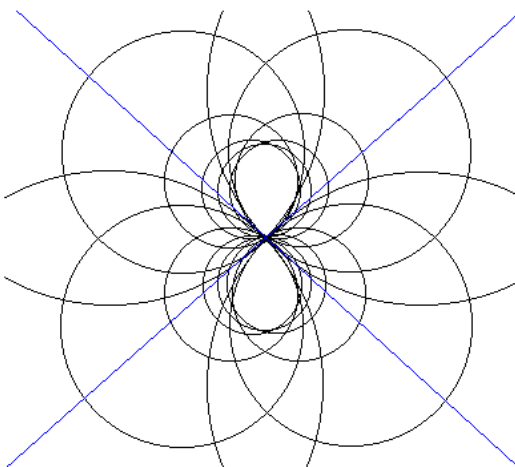


Figure 1

Here two projective ranges on the two blue lines, when joined by circles in Z , yield the figure shown. The construction is well known (e.g. E.H. Lockwood *A Book of Curves*) - although not viewed as above - and generates the lemniscate of Bernoulli.

"I-space" is such a geometry based on points and circles in a fixed point. It was chosen because it contains the lemniscate as a "conic".

As an example of how this geometry works, the following figure shows Pascal's Theorem in I-space:

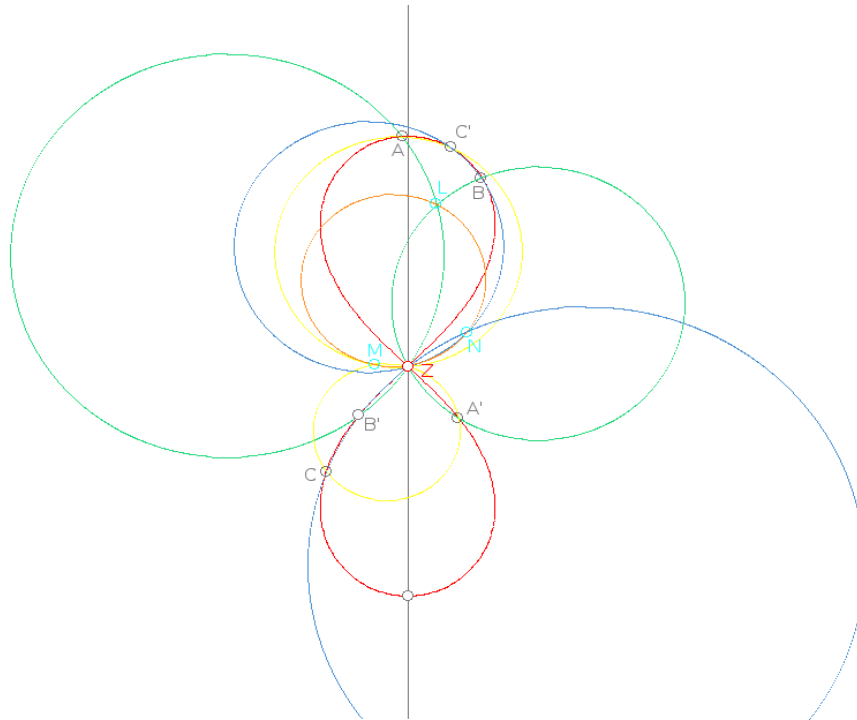


Figure 2

Six points A B C A' B' C' are shown on the lemniscate ("conic"). The two light green circles joining AB' and $A'B$ intersect in L , the two light-blue circles BC' and $B'C$ intersect in N and the two yellow circles joining AC' and $A'C$ meet in M . All joins assume the circles contain Z of course. The three points LMN lie on an orange circle containing Z which is thus a "line" as required by Pascal's Theorem.

We will need to use the concept of "parallel lines" in I-space. These will be two circles that do not meet in a second point apart from Z , which means that they are tangential at Z .

The Problem

A dynamical approach to the problem attempts to calculate a screw-like line along which both the Sun and Earth move, following indications of Rudolf Steiner, which is the projection of a lemniscate from some perspective. George Adams (Ref 2) made a fascinating different approach using an elliptic congruence to project the lemniscate, which inspired the author to try another purely projective approach based on I-space.

At the outset it seemed evident that a careful distinction must be made between “Maya” and “Truth”. Whatever lemniscate approach we adopt we must be able to “save the appearances”, and those appearances are the point of view of Maya. As a beginning we will attempt to save two essential appearances:

- 1) that the fixed stars appear indeed to be fixed;
- 2) that the Sun appears to move continuously round the Zodiac.

Later we must ensure other aspects such as the more or less constant apparent size of the Sun throughout the year, and so forth.

A Solution

Let us first check appearance number one, that the fixed stars are indeed fixed. We take light-rays to follow “straight lines”, which are circles through the point Z (the real base point (RBP) of the I-space concerned). This may accord with Steiner's indication (Ref 5) that light is “elastic” and eventually returns to the Sun after having been emitted. We will first show that taking stars to be fixed locations in space does not work.

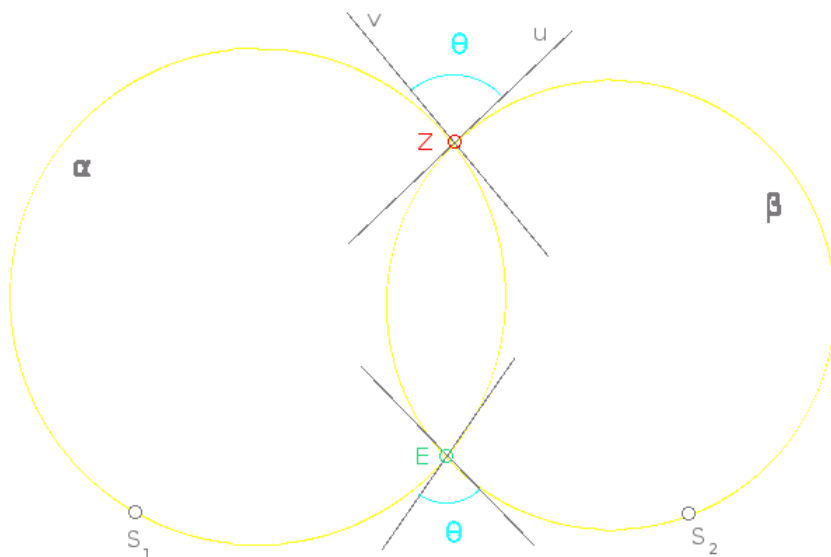
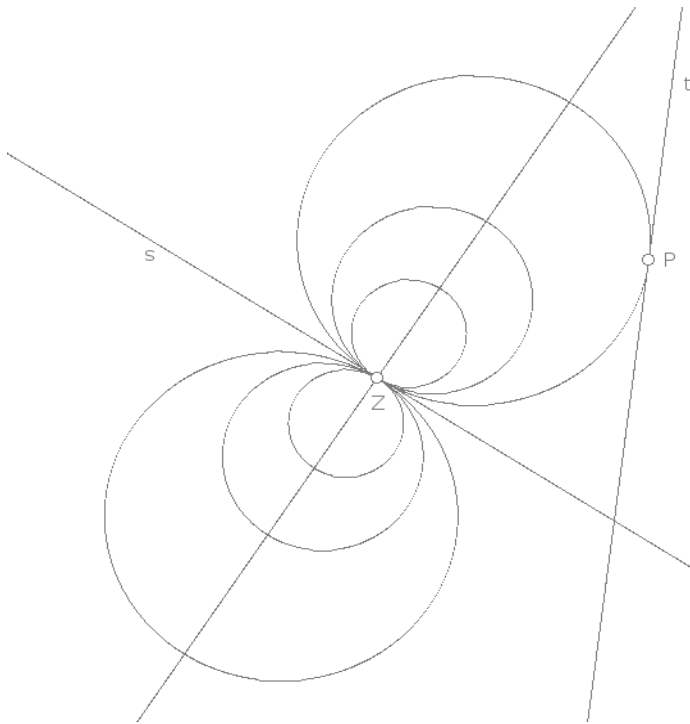


Figure 3

Let there be two fixed stars S_1 and S_2 which will be seen from E (the Earth) by means of light rays α and β which are the unique circles through EZS_1 and EZS_2 respectively, as shown in Figure 3. According to maya we take the stars S_1 and S_2 to appear from the Earth to be in the directions of the tangents at E to the two circles. Those tangents will be at some angle θ which by elementary geometry equals that between the tangents u and v at Z . Now, whatever path E may follow the angle θ must remain constant to save appearance number one. We adopt the simplest metrical definition of angle, which we will accept initially.

The flat pencils (u) and (v) centred on Z , which u and v traverse as E moves, are equal pencils as θ is constant. This generates two pencils of circles in ZS_1 and ZS_2 such that E traverses another circle, which is not what we require.



Thus we reject the assumption that the two stars are at fixed locations in space, and instead adopt Steiner's suggestion that stars only indicate directions in space (Ref 6). One way of defining a direction is by means of a set of parallel lines, which are imaged in I-space by the set of circles tangential to a fixed line s at Z , since two parallel "lines" in I-space are tangential at Z (Figure 4). We say that a fixed star has no location, but is represented by such a set of circles in I-space, including s . At a point P in I-space the star is seen in the direction of the tangent t at P to the circle of the set containing P . To avoid ambiguity a sense will be defined for s and then the sense of the line at P will be that of s carried round the circle.

Figure 4

Now consider two stars represented by the lines S_1 and S_2 through Z , together with their attendant circles, and let the Earth be at E (Figure 5). S_1 and S_2 will be inclined at a fixed angle θ , and the tangents to the two circles containing E , each belonging to one star, will now be at the same angle θ no matter where E is located. Hence the two fixed stars will always appear to be separated by a constant angle no matter what path E follows. This takes care of appearance number one from a two-dimensional perspective.

Now we come to appearance number two. First we will adopt the lemniscate as the mutual Sun/Earth orbit in I-space so that it is a "conic". There are, of course, numerous ways of letting the Sun and Earth move round it, and a reasonable starting point is to make them follow two ranges "in involution" on the "conic".

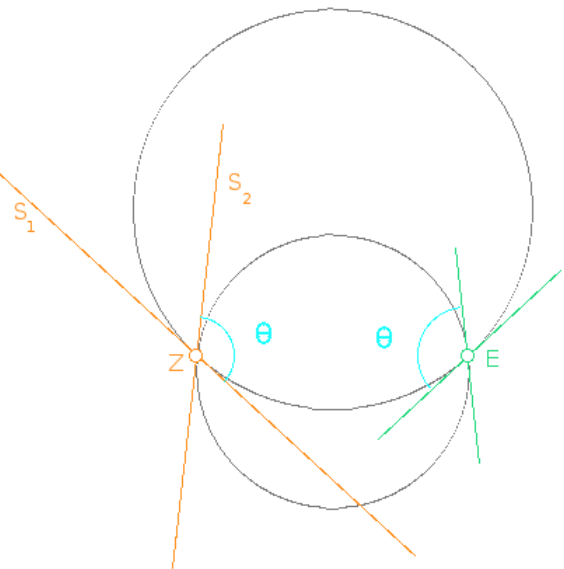


Figure 5

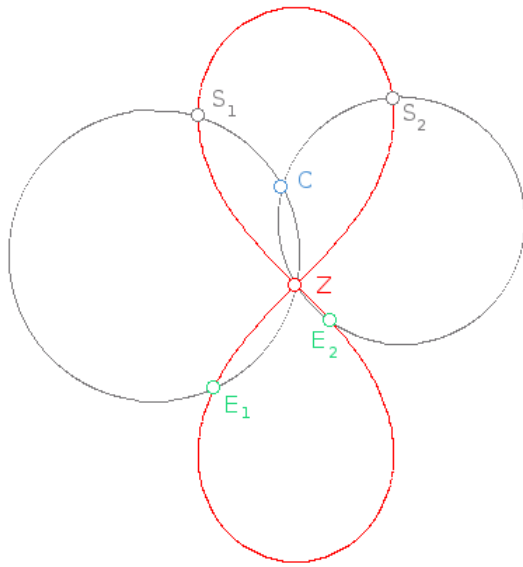


Figure 6

This must be a circling (elliptic) involution to ensure the Sun and Earth never collide. Now such an involution is determined on a conic by the pairs of points in which the lines of a flat pencil meet it, where the centre of that pencil is an interior point. This is shown for I-space in Figure 6 where C is the centre of a pencil of circles. Each circle of the pencil based on C intersects the lemniscate in two points such as S_1 and E_1 defining one mutual relation of the Sun and Earth respectively.

In Figure 7 we show one circle α of the involution for the Sun and Earth at S and E, with a tangent t to the circle at Z. If t rotates uniformly about Z then α will "rotate" about C, remaining tangential to t . The Sun and Earth will then move round the lemniscate smoothly and continuously. Now let there be some fixed star represented by the line s and its tangent circles through Z. The angle θ between s and t will increase uniformly as t rotates. If β is the circle of the fixed star through E, then the angle θ between the tangents to α and β at E will also increase uniformly. Hence the Sun appears to move steadily round the Zodiac, which saves appearance number two.

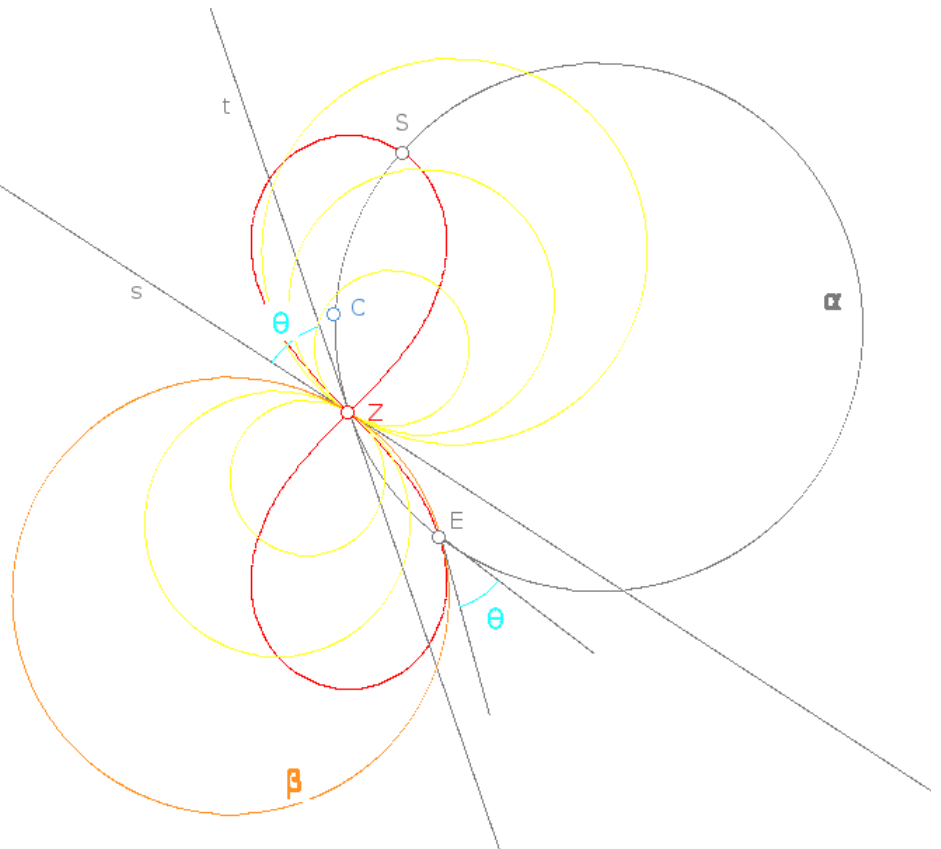


Figure 7

We adopt the following conventions to resolve the ambiguities in direction along a tangent:

1. The lines such as s are given a definite sense, and then the sense round a circle of the star's system is continuous and consistent with that of s at Z ;
2. The Sun is always seen from the Earth in the outward direction from the lemniscate along a tangent such as that at E in Figure 7.

These two conventions ensure that the angle relationships vary consistently round the lemniscate.

Although we have successfully save appearances one and two, in reality the stars possess proper motion which may easily be allowed for by letting their representative lines in Z rotate slowly. Also the Sun varies in speed round the Zodiac in the course of the year. We may allow for this by varying the rate of rotation of t in Figure 7, which is arbitrary, or we choose a space which differs slightly from I -space. We are left with considerable freedom which, although apparently scientifically weak, allows for restricting the choices when the rest of the Solar System is incorporated.

Further Possibilites

Steiner's screw-like spiral may be obtained from this by letting the plane of the lemniscate move parallel to itself. The points occupied by the Sun and Earth will then be in different planes so that they follow the same spiral (Ref. 3), occupying conjugate positions of the involution only when seen in perspective. That the involution is circling is essential in this respect, or a kind of double-spiral would be obtained.

A further suggestion of Steiner's is that whenever the Sun is at Z the Earth is at D_1 or D_2 (Figure 8).

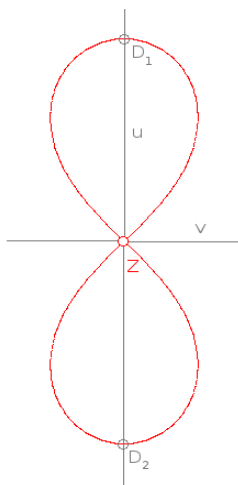


Figure 8

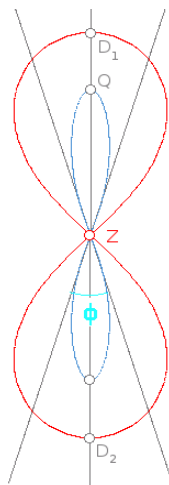


Figure 9

Investigation of this shows that there is a unique pair of projective ranges on the lemniscate with imaginary double-points that meet this condition. They would replace the involution used before. It is necessary to construct a figure-of-eight "conic" shown in Figure 9 which passes through the focal points (such as Q). Its tangents at Z are at an angle $\phi = \tan^{-1} \frac{1}{\sqrt{2}}$. The positions of the Sun and Earth are found from circles through Z tangential to it. The lengthy derivation of this result is omitted as this approach is not yet satisfactory because

appearance number two is not saved: the Sun periodically reverses direction with respect to the Zodiac. A change of metric or some other adjustment would be required to remedy that.

Summary

Several indications of Rudolf Steiner have been adopted in the main approach based on involutions:

that the Sun and Earth move round a mutual lemniscate;
light rays are "elastic", returning to the Sun again;
that the fixed stars indicate directions rather than locations in space.

Appearances one and two are successfully saved, and reasonable adjustments are possible to allow for the proper motions of stars and the varying orbital speed of the Sun and Earth. The approach is two-dimensional to deal with the Sun and Earth in the plane of a lemniscate. Extension to three-dimensions would be necessary to incorporate other planets.

References

1. *Mathematical-Physical Correspondence*, Michaelmas 1975, edited by Stephen Eberhart.
2. George Adams, *The Lemniscatory Ruled Surface in Space and Counterspace* Rudolf Steiner Press 1979.
3. Rudolf Steiner, *The Relation of the Diverse Branches of Natural Science to Astronomy*, lectures given in Stuttgart 1st to 18th January 1921.
4. Rudolf Steiner, *Man: Heiroglyph of the Universe*, lectures given in Dornach 9th April to 16th May 1920.
5. Rudolf Steiner, in *The Evolution of Consciousness*, lecture in Penmaenmawr 31st August 1923.
6. Rudolf Steiner, in *Supersensible Man*, lecture in The Hague 17th November 1923.