Elliptic Congruences and Astronomy

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Introduction

George Adams proposed a beautiful idea for using an elliptic congruence to project a lemniscate into a circle (Ref 1). He intended thus to approach Rudolf Steiner's indications that the planets travel on lemniscates. The present article approaches his idea in a new way that does not require access to his string models, or indeed to complex numbers. It is based on the fact that an elliptic congruence may be mapped onto the real projective plane.

A *congruence* of lines is a set of ∞^2 lines selected by some means from the ∞^4 lines of space, and in particular a *linear congruence* is such that every plane of space contains one of its lines, and every point of space also contains just one of its lines. The clearest example is the set of ∞^2 lines intersecting two given skew *bearer lines* v and w. For given any point V on v, there is a flat pencil of ∞^1 lines through V meeting w, and hence ∞^2 lines meeting both v and w when V moves along v. Given an arbitrary point P, just one line through P intersects both v and w, which is the point in which w meets the plane (P,v). Also, given an arbitrary plane α, the bearer lines v and w meet it in the points V and W, say, so the line VW is the unique line of the congruence lying in α. Hence the congruence is linear and is referred to as a *hyperbolic congruence*. It is possible for v and w to coincide in which case the congruence is termed *parabolic*, determined by a projectivity between the points and planes of v. Given any point P on v there is a corresponding plane α which contains a flat pencil of lines in P and α, and thus ∞^2 lines in total when all corresponding point/plane pairs are taken. This is also known as a *parabolic strip*.

A congruence need not have real bearer lines, in which case it is termed an *elliptic congruence*. The clearest example is what George Adams termed the *mother form*, which is readily visualised as follows. Take a vertical line ℓ as axis and all the reguli having that axis in common and with a common "waist plane" ω and *modulus*. The latter is the fixed number d in the equation $d=r \tan\theta$, where at a radius r from ℓ in ω the rulers of the corresponding regulus are at an angle θ to ω. The modulus d is the value of r when $\theta=45^o$. This is illustrated below.

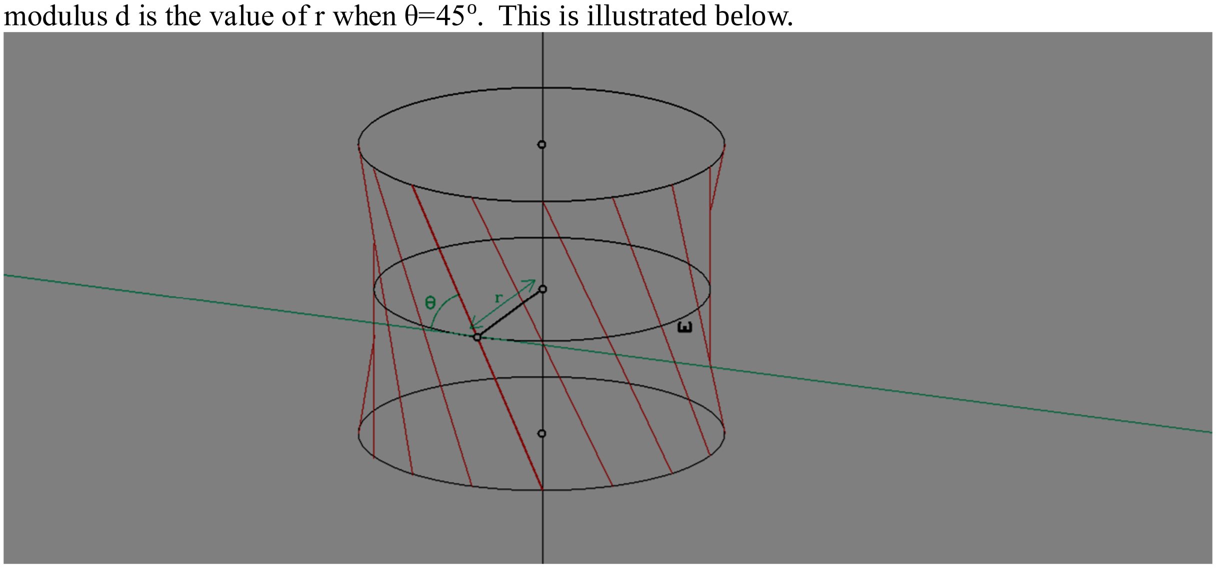

Figure &1

Thus as r increases θ decreases, so the rulers tend towards the horizontal with increasing r, which illustrates the relationship between the reguli making up the congruence. The vertical axis ℓ and the line in which ω meets the plane at infinity are degenerate reguli of the pencil.

In counterspace terms, tanθ is a measure of the turn of a plane in from the plane at infinity (Ref. 2), so r tanθ being constant shows a fixed relationship between radial distance and turn of the tangent planes where the rulers are at 45º.

There are ∞^1 lines in each such regulus and ∞^1 such reguli yielding ∞^2 lines in all i.e. a congruence. Given a general point P, it lies on one of the reguli and one ruler of that regulus lies in P. On the other hand a general plane α is tangential to just one of the reguli $\mathcal{R}$, containing the ruler of $\mathcal{R}$ in α as the only line of the congruence in α. This is illustrated below.

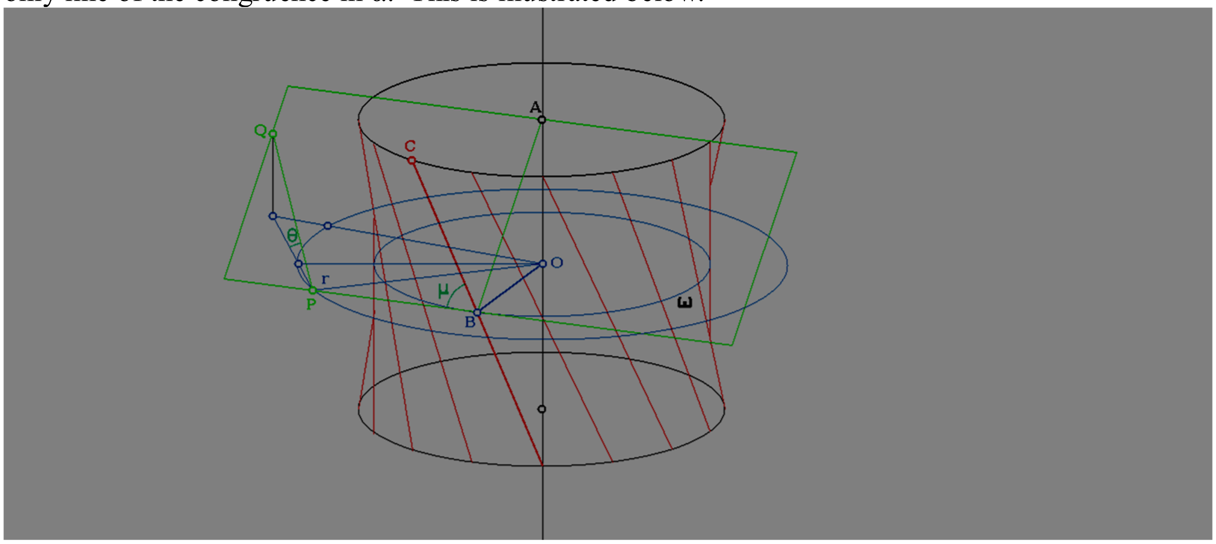

Figure 2

The plane α is shown in green intersecting the axis ℓ in A and the waist plane in a line BP whose perpendicular distance from O is OB. The line of the regulus through B is BC orthogonal to OB, which clearly cannot not lie in α. If we vary a point P along the line BP the radius r increases as it moves away from B, and the angle θ decreases, so at a unique position of P, PQ will lie in α for r tanθ = d. Note that PQ is horizontal when r=∞ so it must pass from one side of α to the other at some point. For a conjugate regulus P would move in the opposite direction.

Thus the congruence is linear since a point contains one line of the congruence, as does a plane. It has been treated metrically above but is in fact purely projective, the metric view being merely an aid to the imagination. Projectively, the set of reguli form a *pencil*.

Embedding a Congruence in a Bundle

1. An interesting way of viewing a congruence is to take an arbitrary point B containing a bundle of planes. There is one line of a linear congruence in every such plane, and every line of the congruence determines a plane of the bundle, so the whole congruence is contained in the bundle. This approach affords an entirely non-algebraic approach to some cubic surfaces and envelopes (which will not be described here), and to an alternative generation of congruences and complexes. Here we are concerned with linear congruences.

2. Given two bundles B_1 B_2 each containing the same congruence, there is clearly a regulus of lines in the axial pencil B_1B_2 unless B_1B_2 is a bearer line i.e. there is a regulus of the congruence in every line of space.

3. If a bundle in B contains a congruence *C* then a self-collineation $\mathbb{C}$ of the bundle induces a transformation among the lines of the congruence. There will be an invariant trihedron of the bundle

containing three self-corresponding lines of *C* (wrt $\mathbb{C}$), which form the "invariant triangle" $\mathfrak{I}$ of *C* analogous to the invariant triangle of a planar two-dimensional self-collineation. There will be "path developables" of the lines of the congruence leading from one invariant line to another.

4. *C* can be mapped onto the real projective plane as follows. If the lines of *C* are analogous to *points*, we must define its *lines*. Italics will be used to denote the entities of the congruence mapped onto the real projective plane. There are ∞^2 lines in B which act as the axes of axial pencils of planes in B, each such pencil containing a regulus of *C* (c.f. 2 above); two such pencils have a common plane containing a line of *C* (a *point*), and two planes determine the axis of a pencil containing a regulus. Thus those reguli of *C* may act as *lines*, and the axioms for two dimensional projective geometry are satisfied for *lines* and *points*. Then the lines forming $\mathfrak{I}$ are its *vertices*, and its *invariant lines* are the three reguli determined by pairs of planes through pairs of its lines each meeting in an axis in B.

5. This is a remarkable example of a realisation of two-dimensional projective geometry with no "line at infinity", where whole *lines* are closed in the form of reguli properly imaging projective lines. Thus a linear congruence may be mapped onto the real projective plane in this way.

6. What are its *conics*? They are imaged *pointwise* by quadric cones in B containing a line of *C* in each tangent plane. Such a cone contains a ruled surface of the fourth order. For, given a ruled surface S in the planes of a quadric cone C_q in B, a general line q is intersected by the lines of a regulus $\mathcal{R}$ of *C*, and those lines project a quadric wrap C_r in B. C_r and C_q possess at most four common tangent planes in general, containing four lines common to S and $\mathcal{R}$. Since q meets those four lines it meets S in four points and so S is in general a fourth order ruled surface. If S has a node (i.e. crosses itself in one of its rulers) then two planes of C_q must meet in a line of *C*. But only one pair can do that as there is only one line ℓ of *C* in B, so neither bi-nodal nor tri-nodal ruled surfaces are possible (for real nodes, that is). It is interesting that this is not reflected in the conic imaged by S, unless that degenerates into two *lines*. But that requires C_q to consist of two reguli in axial pencils, which is of course trivially possible, but in general C_q is a proper quadric cone. Furthermore, if ℓ lies inside the cone then S has no node (e.g. lines in a Cassini oval) whereas if it lies outside the cone then S necessarily has a node as there are two distinct tangent planes to C_q from ℓ.

7. A special case concerns a regulus $\mathcal{R}$ of *C* which meets a regulus that is a *line* if they possess common rulers (*points*), and generally they may possess at most two common rulers. Thus $\mathcal{R}$ is a *conic* as it is intersected by a *line* in at most two *points*. (They <u>can</u> meet in two points as two rulers of a *line p* may be chosen, and then a third line of *C* not in that *line* which will determine a regulus of *C* that is not a *line*, met by *p* in two *points*). A quadric cone in B is determined by five of its planes, which accords with the fact that a conic is determined by five of its points. But the above reguli acting as *conics* are determined by only three *points*, suggesting they are *circles*. But that is not possible as there is no implied metric in the real projective plane which could distinguish circles. Thus they must in fact be *general conics*. Their lines subtend a conical wrap in B which thus contains a regulus of *C*. Strictly speaking the line ℓ in B counted twice is part of the *conic*.

8. In particular S may be lemniscatory since a lemniscate has one real node, and we then have a lemniscatory ruled surface acting as a *conic*. If we select any plane α then the lines of S meet it in a fourth-order figure. If α contains a line of S the residual intersection will be cubic, and if α contains ℓ then it effectively contains two lines of S so the residual intersection is a conic. This is the case George Adams found (Ref 1) where the congruence projects a lemniscate into a conic, given that S is a ruled lemniscatory surface. Note that the conic lies in a plane containing ℓ, mirroring the fact that in Adam's case the conic lay in a plane of the vertical axis of his mother form, the latter corresponding to our ℓ. He chose the "waist plane" of the mother form for the lemniscate Λ so that no line of S is contained, giving a section of the fourth order. Any plane of B that does not cut the C_q containing S will contain a quartic section of the ruled surface. Thus if ℓ is vertical and C_q is a circular cone with

an axis q orthogonal to ℓ, and we take the plane β containing that axis and orthogonal to ℓ, the situation is symmetrical in β giving a proper lemniscate. The lines of *C* through that are at various inclinations and will meet the plane α in ℓ orthogonal to q in a conic, which is a circle together with ℓ counted twice (when Λ is a lemniscate of Bernoulli). Note that the rulers in the extremities of the lemniscate axis only meet α where the lemniscate meets it, so the circle must meet α there too i.e. the circle and α intersect in the extremities of the lemniscate. Of course the congruence must be elliptic to mirror the work of Adams, and it may manifest metrically for our ordinary consciousness.

9. To construct the above we use the formula d=r tanθ already considered earlier. For a lemniscate, r is the radial distance from ℓ to a point P on it, which gives θ from the selected value of d. The ruler (congruence line) is drawn through P orthogonal to OP and at the angle θ to the horizontal, to meet the vertical plane in a point of the circle. A computer program implementing this gave the following result:

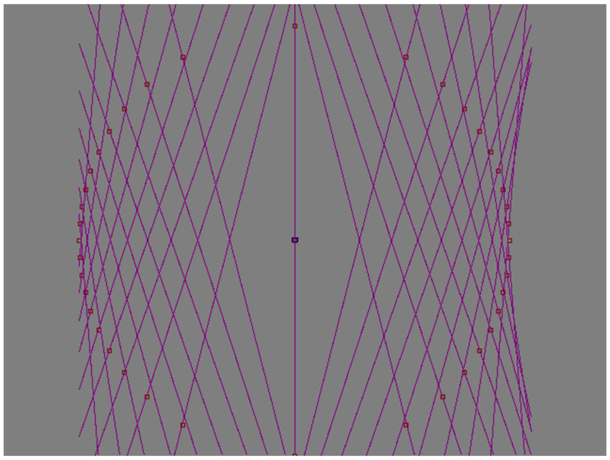

Figure 3

This shows the circle calculated as above, thus verifying the theory, together with the congruence lines projecting the lemniscate into it. The lemniscate is in the horizontal plane through the centre normal to the plot. The following diagram is a tilted perspective view of the above:

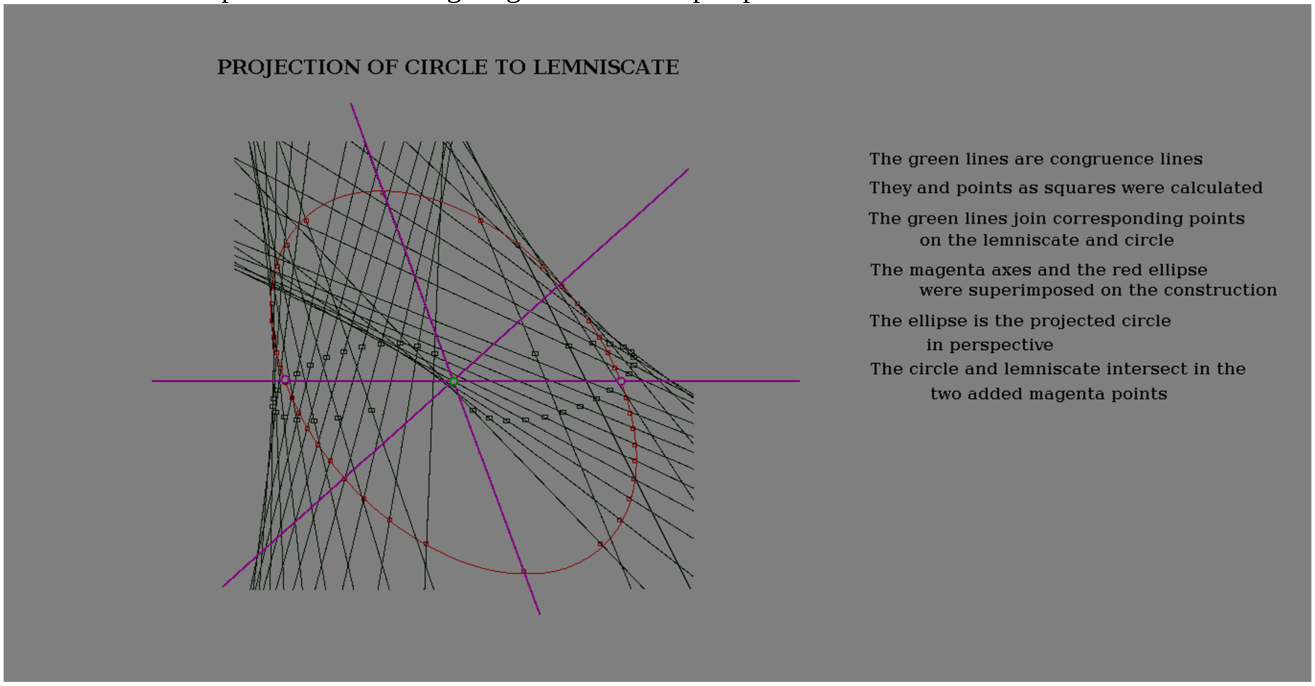

Figure 4

The following diagram based on this clarifies the projection:

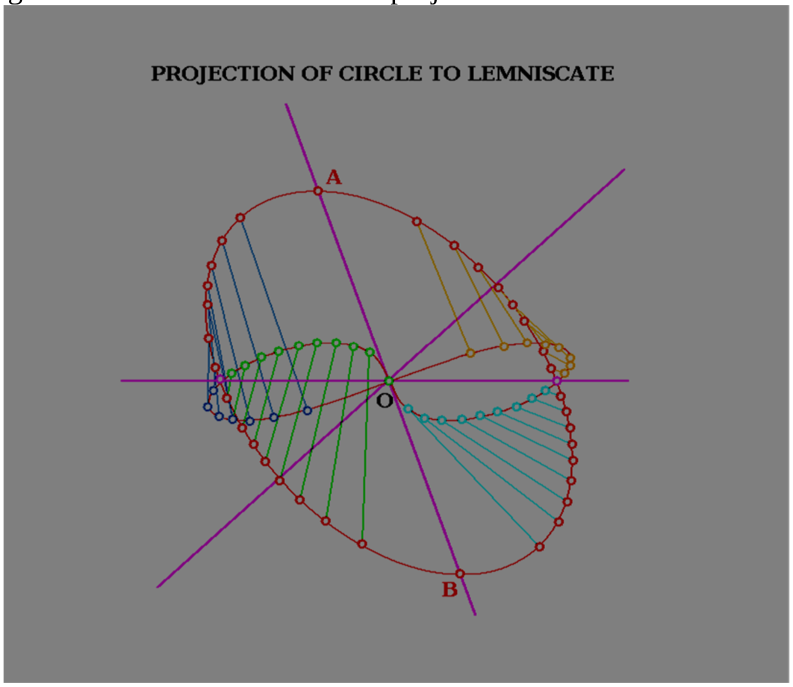

Figure 5

Application to Astronomy

10. But how can this really explain Rudolf Steiner's astronomical indications? If the planets actually travel on lemniscates then our view of them is based not on a standard projection of light rays but on a projection via an elliptic congruence, so that they appear to travel on nearly circular ellipses. It is easy to accommodate ellipses in the above scheme. Then the apparent Ecliptic would contain ellipses such as above for the various planets, and another orthogonal *Ecliptic* would contain the more general lemniscates. Based on that first approximation, the inclinations of orbits can easily be included. The curious result is that the lemniscates have their nodes in or very near the Sun, which suggests an alternative interpretation:

that the lemniscates are in the etheric realm, projected onto the physical elliptic orbits by an elliptic congruence mediating between space and counterspace. Then at the etheric level the fact that the nodes are in the Sun need not be a problem, and indeed may be significant for the way in which the planets relate to the Sun spiritually. For then they pass through the etheric Sun twice on every orbit. The times of such events may be significant for astrosophy.

Conclusion

The embedding of an elliptic congruence in a bundle of planes enables that congruence to be mapped onto the real projective plane. Then lemniscatory ruled surfaces may act as *conics* in the mapping, giving an alternative approach to the work of George Adams. The application to astronomy seems best to regard the lemniscates as being in the etheric realm, and opens up a new perspective on how the planets relate to the Sun spiritually.

References

1. Adams, George, *The Lemniscatory Ruled Surface in Space and Counterspace*, Rudolf Steiner Press, London 1979, translated by Stephen Eberhart from a letter in 1936 in German to Dr. Elizabeth Vreede.

2. Thomas, N.C., *Science Between Space and Counterspace,* New Science Books, London 1999, reprinted in 2008 by Temple Lodge Press.